Enumeration of curves with two singular points

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\textbf{Abstract}

In this paper we obtain an explicit formula for the number of curves in $\mathbb{P}^2$, of degree $d$, passing through $(d(d+3)/2-(k+1))$ generic points and having one node and one codimension $k$ singularity, where $k$ is at most 6. Our main tool is a classical fact from differential topology: the number of zeros of a generic smooth section of a vector bundle $V$ over $M$, counted with a sign, is the Euler class of $V$ evaluated on the fundamental class of $M$.

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1. Introduction

Enumeration of singular curves in $\mathbb{P}^2$ (complex projective space) is a classical problem in algebraic geometry. In the paper [1], we used purely topological methods to answer the following question:

\textbf{Question 1.1.} How many degree-$d$ curves are there in $\mathbb{P}^2$, passing through $(d(d+3)/2-k)$ generic points and having one singularity of codimension $k$, where $k$ is at most 7?

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In this paper, we extend the methods used in [1] to enumerate curves with two singular points. More precisely, we obtain an explicit answer for the following question:

**Question 1.2.** How many degree-$d$ curves are there in $\mathbb{P}^2$, passing through $(d(d + 3)/2 - (k + 1))$ generic points, having one node and one singularity of codimension $k$, where $k$ is at most 6?

Let us denote the space of curves of degree $d$ in $\mathbb{P}^2$ by $\mathcal{D}$. It follows that $\mathcal{D} \cong \mathbb{P}^{d_d}$, where $d_d = d(d + 3)/2$. Let $\gamma_{\mathbb{P}^2} \to \mathbb{P}^2$ be the tautological line bundle. A homogeneous polynomial $f$, of degree $d$ and in 3 variables, induces a holomorphic section of the line bundle $\gamma_{\mathbb{P}^2} \to \mathbb{P}^2$. If $f$ is non-zero, then we will denote its equivalence class in $\mathcal{D}$ by $\hat{f}$. Similarly, if $p$ is a non-zero vector in $\mathbb{C}^3$, we will denote its equivalence class in $\mathbb{P}^2$ by $\hat{p}$.1

**Definition 1.3.** Let $\hat{f} \in \mathcal{D}$ and $\hat{p} \in \mathbb{P}^2$. A point $\hat{p} \in f^{-1}(0)$ is of singularity type $\mathcal{A}_k$, $\mathcal{D}_k$, $\mathcal{E}_6$, $\mathcal{E}_7$, $\mathcal{E}_8$ or $\mathcal{X}_8$ if there exists a coordinate system $(x,y) : (U, \hat{p}) \to (\mathbb{C}^2, 0)$ such that $f^{-1}(0) \cap U$ is given by

$$
\begin{align*}
\mathcal{A}_k : & y^2 + x^{k+1} = 0 \quad k \geq 0, \\
\mathcal{D}_k : & y^2 x + x^{k-1} = 0 \quad k \geq 4, \\
\mathcal{E}_6 : & y^3 + x^4 = 0, \\
\mathcal{E}_7 : & y^3 + yx^3 = 0, \\
\mathcal{E}_8 : & y^3 + x^5 = 0, \\
\mathcal{X}_8 : & x^4 + y^4 = 0.
\end{align*}
$$

In more common terminology, $\hat{p}$ is a smooth point of $f^{-1}(0)$ if it is a singularity of type $\mathcal{A}_0$; a simple node if its singularity type is $\mathcal{A}_1$; a cusp if its type is $\mathcal{A}_2$; a tacnode if its type is $\mathcal{A}_3$; a triple point if its type is $\mathcal{D}_4$; and a quadruple point if its type is $\mathcal{X}_8$.

We have several results (cf. Eqs. (3.2)–(3.13), Section 3) which can be summarized collectively as our main result. Although (3.2)–(3.13) may appear as equalities, the content of each of these equations is a theorem. These are the basic identities which lead to the following:

**MAIN THEOREM.** Let $\mathcal{X}_k$ be a singularity of type $\mathcal{A}_k$, $\mathcal{D}_k$ or $\mathcal{E}_k$. Denote by $N(\mathcal{A}_1, \mathcal{X}_k, n)$ the number of degree-$d$ curves in $\mathbb{P}^2$ that pass through $\delta_d - (k + 1 + n)$ generic points having one $\mathcal{A}_1$ node and one singularity of type $\mathcal{X}_k$ at the intersection of $n$ generic lines. Then there is an explicit formula for $N(\mathcal{A}_1, \mathcal{X}_k, n)$ if $k + 1 \leq 7$, provided $d \geq C_{\mathcal{A}_1, \mathcal{X}_k}$ where

$$
C_{\mathcal{A}_1, \mathcal{A}_k} = k + 3, \quad C_{\mathcal{A}_1, \mathcal{D}_k} = k + 1, \quad C_{\mathcal{A}_1, \mathcal{E}_6} = 6.
$$

**Remark 1.4.** The explicit formulas are given in Section 3.1. Eqs. (3.2)–(3.13) are used recursively to obtain the explicit formulas.

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1 In this paper we will use the symbol $\hat{A}$ to denote the equivalence class of $A$ instead of the standard $[A]$. This will make some of the calculations in Section 6 easier to read.
Remark 1.5. Note that $\mathcal{N}(A_1X_k, n)$ is zero if $n > 2$, since three or more generic lines do not intersect at a common point. Moreover, $\mathcal{N}(A_1X_k, 2)$ is the number of curves, of degree $d$, that pass through $\delta_d - (k + 3)$ generic points having one $A_1$-node and one singularity of type $X_k$ lying at a given fixed point (since the intersection of two generic lines is a point).

In [1], we obtained an explicit formula for $\mathcal{N}(X_k, n)$, the number of degree-$d$ curves in $\mathbb{P}^2$ that pass through $\delta_d - (k + n)$ generic points having one singularity of type $X_k$ at the intersection of $n$ generic lines. We extend the methods applied in [1] to obtain an explicit formula for $\mathcal{N}(A_1X_k, n)$.

The numbers $\mathcal{N}(A_1X_k, 0)$ till $k+1 \leq 7$ have also been computed by Maxim Kazarian [4] using different methods. Furthermore, the numbers $\mathcal{N}(A_1A_1, 0)$, $\mathcal{N}(A_1A_2, 0)$ and $\mathcal{N}(A_1D_4, 0)$ have also been computed by Dmitry Kerner [5] using different methods. Our results for $n = 0$ agree with theirs. The bound $d \geq C_{A_1X_k}$ is imposed to ensure that the relevant bundle sections are transverse to the zero set.\(^2\) The formulas for $\mathcal{N}(A_1A_1, n)$ and $\mathcal{N}(A_1A_2, n)$ also appear in [7]. We extend the methods applied by the author to obtain the remaining formulas.

To the best of our knowledge, the formulas obtained in this paper for $\mathcal{N}(A_1X_k, n)$ for $n > 0$ are new.\(^3\)

2. Overview

Our main tool will be the following well known fact from topology (cf. [3, Proposition 12.8]).

**Theorem 2.1.** Let $V \to X$ be a vector bundle over a manifold $X$. Then the following are true:

1. A generic smooth section $s : X \to V$ is transverse to the zero set.
2. Furthermore, if $V$ and $X$ are oriented with $X$ compact then the zero set of such a section defines an integer homology class in $X$, whose Poincaré dual is the Euler class of $V$. In particular, if the rank of $V$ is same as the dimension of $X$, then the signed cardinality of $s^{-1}(0)$ is the Euler class of $V$, evaluated on the fundamental class of $X$, i.e.,

$$|\pm s^{-1}(0)| = \langle e(V), [X] \rangle.$$  

**Remark 2.2.** Let $X$ be a compact, complex manifold, $V$ a holomorphic vector bundle and $s$ a holomorphic section that is transverse to the zero set. If the rank of $V$ is same as

\(^2\) However, this bound is not the optimal bound.

\(^3\) Except for $\mathcal{N}(A_1A_1, n)$ and $\mathcal{N}(A_1A_2, n)$ which appears in [7].
the dimension of $X$, then the signed cardinality of $s^{-1}(0)$ is same as its actual cardinality (provided $X$ and $V$ have their natural orientations).

However, for our purposes, the requirement that $X$ is a smooth manifold is too strong. We will typically be dealing with spaces that are smooth but have non-smooth closure. The following result is a stronger version of Theorem 2.1, that applies to singular spaces, provided the set of singular points is of real codimension two or more.

**Theorem 2.3.** Let $M \subset \mathbb{P}^N$ be a smooth, compact algebraic variety and $X \subset M$ a smooth subvariety, not necessarily closed. Let $V \to M$ be an oriented vector bundle, such that the rank of $V$ is same as the dimension of $X$. Then the following are true:

1. The closure of $X$ is an algebraic variety and defines a homology class.
2. The zero set of a generic smooth section $s : M \to V$ intersects $X$ transversely and does not intersect $\overline{X} - X$ anywhere.
3. The number of zeros of such a section inside $X$, counted with signs, is the Euler class of $V$ evaluated on the homology class $[\overline{X}]$, i.e.,

$$|\pm s^{-1}(0) \cap \overline{X}| = |\pm s^{-1}(0) \cap X| = \langle e(V), [\overline{X}] \rangle.$$

**Remark 2.4.** All the subsequent statements we make are true provided $d$ is sufficiently large. The precise bound on $d$ is given in Section 5. Although the results of this paper are an extension of [1], our aim has been to keep this paper self-contained. Ideally, a reader not familiar with [1] should have no difficulty following this paper.

We will now explain our strategy to compute $\mathcal{N}(\mathcal{A}_1 \mathcal{X}_k, n)$. The strategy is very similar to that of computing $\mathcal{N}(\mathcal{X}_k, n)$, which was the content of [1]. Let $X_1$ and $X_2$ be two subsets of $\mathcal{D} \times \mathbb{P}^2$. Then we define

$$X_1 \cap X_2 := \{ (\hat{f}, \hat{p}, \tilde{p}_2) \in \mathcal{D} \times \mathbb{P}^2 \times \mathbb{P}^2 : (\hat{f}, \hat{p}_1) \in X_1, (\hat{f}, \tilde{p}_2) \in X_2, \hat{p}_1 \neq \tilde{p}_2 \}.$$

Next, given a subset $X$ of $\mathcal{D} \times \mathbb{P}^2$ we define

$$\Delta X := \{ (\hat{f}, \hat{p}, \hat{p}) \in \mathcal{D} \times \mathbb{P}^2 \times \mathbb{P}^2 : (\hat{f}, \hat{p}) \in X \}.$$

Similarly, let $X_1$ and $X_2$ be two subsets of $\mathcal{D} \times \mathbb{P}^2$ and $\mathcal{D} \times \mathbb{T} \mathbb{P}^2$ respectively. Then we define

$$X_1 \cap X_2 := \{ (\hat{f}, \tilde{p}_1, l_{\tilde{p}_2}) \in \mathcal{D} \times \mathbb{P}^2 \times \mathbb{T} \mathbb{P}^2 : (\hat{f}, \tilde{p}_1) \in X_1, (\hat{f}, l_{\tilde{p}_2}) \in X_2, \tilde{p}_1 \neq \tilde{p}_2 \}.$$

Finally, given a subset $X$ of $\mathcal{D} \times \mathbb{T} \mathbb{P}^2$ we define

$$\Delta X := \{ (\hat{f}, \tilde{p}, l_{\tilde{p}}) \in \mathcal{D} \times \mathbb{P}^2 \times \mathbb{T} \mathbb{P}^2 : (\hat{f}, l_{\tilde{p}}) \in X \}.$$

The following result is clear from the definition of closure.
Lemma 2.5. We have the following equality of sets

\[
\tilde{X}_1 \circ \tilde{X}_2 = \tilde{X}_1 \circ X_2 = \tilde{X}_1 \circ \tilde{X}_2.
\]

Given a singularity \( \tilde{x}_k \), we also denote by \( x_k \), the space of curves of degree \( d \) with a marked point \( \tilde{p} \) such that the curve has a singularity of type \( x_k \) at \( \tilde{p} \). Similarly, \( A_1 \circ x_k \) is the space of degree-\( d \) curves with two distinct marked points \( \tilde{p}_1 \) and \( \tilde{p}_2 \) such that the curve has a node at \( \tilde{p}_1 \) and a singularity of type \( x_k \) at \( \tilde{p}_2 \). Note that except when \( x_k = A_1 \), the space \( A_1 \circ x_k \) is the fibre product \( A_1 \times \tilde{x}_k \).

Let \( \tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_{\delta_d-(k+1+n)} \) be \( \delta_d - (k + 1 + n) \) generic points in \( \mathbb{P}^2 \) and \( L_1, L_2, \ldots, L_n \) be \( n \) generic lines in \( \mathbb{P}^2 \). Define the following sets

\[
H_i := \{ f \in D : f(p_i) = 0 \}, \quad H^*_i := \{ f \in D : f(p_i) = 0, \ \nabla f|_{p_i} \neq 0 \},
\]

\[
\hat{H}_i := H_i \times \mathbb{P}^2 \times \mathbb{P}^2, \quad \hat{H}^*_i := H^*_i \times \mathbb{P}^2 \times \mathbb{P}^2 \quad \text{and} \quad \hat{L}_i := D \times \mathbb{P}^2 \times L_i. \tag{2.1}
\]

By definition, our desired number \( \mathcal{N}(A_1, x_k, n) \) is the cardinality of the set

\[
\mathcal{N}(A_1, x_k, n) := |A_1 \circ x_k \cap \hat{H}_1 \cap \cdots \cap \hat{H}_{\delta_d-(k+1+n)} \cap \hat{L}_1 \cap \cdots \cap \hat{L}_n|. \tag{2.2}
\]

Let us now clarify an important point to avoid confusion: as per our notation, the number \( \mathcal{N}(A_1, A_1, 0) \) is the number of degree-\( d \) curves through \( \delta_d - 2 \) generic points having two ordered nodes. To find the corresponding number of curves where the nodes are unordered, we have to divide by 2.

We will now describe the various steps involved to obtain an explicit formula for \( \mathcal{N}(A_1, x_k, n) \).

**Step 1.** Our first observation is that if \( d \) is sufficiently large then \( A_1 \circ x_k \) is a smooth algebraic variety and its closure defines a homology class.

**Lemma 2.6.** (Cf. Section 5.) The space \( A_1 \circ x_k \) is a smooth subvariety of \( D \times \mathbb{P}^2 \times \mathbb{P}^2 \) of dimension \( \delta_d - k \).

**Step 2.** Next we observe that if the points and lines are chosen generically, then the corresponding hyperplanes and lines defined in (2.1) will intersect our space \( A_1 \circ x_k \) transversely. Moreover, they would not intersect any extra points in the closure.

**Lemma 2.7.** Let \( \tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_{\delta_d-(k+1+n)} \) be \( \delta_d - (k + 1 + n) \) generic points in \( \mathbb{P}^2 \) and \( L_1, L_2, \ldots, L_n \) be \( n \) generic lines in \( \mathbb{P}^2 \). Let \( \hat{H}_i, \hat{H}_i^* \) and \( \hat{L}_i \) be as defined in (2.1). Then

\[
A_1 \circ x_k \cap \hat{H}_1 \cap \cdots \cap \hat{H}_{\delta_d-(k+n+1)} \cap \hat{L}_1 \cap \cdots \cap \hat{L}_n
\]

\[
= A_1 \circ x_k \cap \hat{H}_1^* \cap \cdots \cap \hat{H}_{\delta_d-(k+n+1)}^* \cap \hat{L}_1 \cap \cdots \cap \hat{L}_n
\]

and every intersection is transverse.
We omit the details of the proof; it follows from an application of the families transversality theorem and Bertini’s theorem. The details of this proof can be found in [2].

**Notation 2.8.** Let $\gamma_D \to D$ and $\gamma_{\mathbb{P}^2} \to \mathbb{P}^2$ denote the tautological line bundles. If $c_1(V)$ denotes the first Chern class of a vector bundle then we set

$$y := c_1(\gamma_D^*) \in H^2(D; \mathbb{Z}), \quad a := c_1(\gamma_{\mathbb{P}^2}^*) \in H^2(\mathbb{P}^2; \mathbb{Z}).$$

As a consequence of Lemma 2.7 we obtain the following fact:

**Lemma 2.9.** The number $\mathcal{N}(\mathcal{A}_1\mathcal{X}_k, n)$ is given by

$$\mathcal{N}(\mathcal{A}_1\mathcal{X}_k, n) = \langle (\pi_D^* y)^{\delta_d -(n+k+1)} (\pi_2^* a)^{n}, [\mathcal{A}_1 \circ \mathcal{X}_k] \rangle$$

where $\pi_D, \pi_1, \pi_2 : D \times \mathbb{P}^2_1 \times \mathbb{P}^2_2 \to D, \mathbb{P}^2_1, \mathbb{P}^2_2$ are the projection maps.

**Proof.** This follows from Theorem 2.3 and Lemma 2.7. \qed

As explained in [1], the space $\mathcal{X}_k$ is not easy to describe directly and hence computing $\mathcal{N}(\mathcal{X}_k, n)$ directly is not easy. As a result we define another space $\mathcal{P}\mathcal{X}_k \subset D \times \mathbb{P}\mathbb{T}\mathbb{P}^2$. This is the space of curves $\tilde{f}$, of degree $d$, with a marked point $\tilde{p} \in \mathbb{P}^2$ and a marked direction $l_{\tilde{p}} \in \mathbb{P}\mathbb{T}_{\tilde{p}}\mathbb{P}^2$, such that the curve $f$ has a singularity of type $\mathcal{X}_k$ at $\tilde{p}$ and certain directional derivatives vanish along $l_{\tilde{p}}$, and certain other derivatives do not vanish. To take a simple example, $\mathcal{P}\mathcal{A}_2$ is the space of curves $\tilde{f}$ with a marked point $\tilde{p}$ and a marked direction $l_{\tilde{p}}$ such that $f$ has an $\mathcal{A}_2$-node at $\tilde{p}$ and the Hessian is degenerate along $l_{\tilde{p}}$, but the third derivative along $l_{\tilde{p}}$ is non-zero. It turns out that this space is much easier to describe. We have defined $\mathcal{P}\mathcal{X}_k$ in Section 4.3. Similarly, instead of dealing with the space $\mathcal{A}_1 \circ \mathcal{X}_k$, we deal with the space $\mathcal{A}_1 \circ \mathcal{P}\mathcal{X}_k$.

**Step 3.** Next we observe that since $\mathcal{A}_1 \circ \mathcal{P}\mathcal{X}_k$ is described locally as the vanishing of certain sections that are transverse to the zero set, these are smooth algebraic varieties.

**Lemma 2.10.** (Cf. Section 5.) The space $\mathcal{A}_1 \circ \mathcal{P}\mathcal{X}_k$ is a smooth subvariety of $D \times \mathbb{P}^2 \times \mathbb{P}\mathbb{T}\mathbb{P}^2$ of dimension $\delta_d -(k+1)$.

**Notation 2.11.** Let $\tilde{\gamma} \to \mathbb{P}\mathbb{T}\mathbb{P}^2$ be the tautological line bundle. The first Chern class of the dual will be denoted by $\lambda := c_1(\tilde{\gamma}^*) \in H^2(\mathbb{P}\mathbb{T}\mathbb{P}^2; \mathbb{Z})$.

Lemma 2.10 now motivates the following definition:

**Definition 2.12.** We define the number $\mathcal{N}(\mathcal{A}_1\mathcal{P}\mathcal{X}_k, n, m)$ as
\[ \mathcal{N}(A_1 \mathcal{P} x_k, n, m) := \langle \pi_{\mathcal{D}}^* y^{\delta_4-(k+n+m+1)} \pi_2^* a^n \pi_2^* \lambda^m, [\mathcal{A}_1 \circ \mathcal{P} x_k] \rangle, \]  

(2.3)

where \( \pi_{\mathcal{D}}, \pi_1, \pi_2 : \mathcal{D} \times \mathbb{P}_1^2 \times \mathbb{P} \mathcal{T} \mathcal{P}_2 \to \mathcal{D}, \mathbb{P}_1^2, \mathbb{P} \mathcal{T} \mathcal{P}_2 \) are the projection maps.

The next lemma relates the numbers \( \mathcal{N}(A_1 \mathcal{P} x_k, n, m) \) and \( \mathcal{N}(A_1 x_k, n) \).

**Lemma 2.13.** The projection map \( \pi : A_1 \circ \mathcal{P} x_k \to A_1 \circ x_k \) is one to one if \( x_k = A_k, D_k, E_6, E_7 \) or \( E_8 \) except for \( x_k = D_4 \) when it is three to one. In particular,

\[ \mathcal{N}(A_1 x_k, n) = \mathcal{N}(A_1 \mathcal{P} x_k, n, 0) \]  

(2.4)

if \( x_k \neq D_4 \) and \( \mathcal{N}(A_1 D_4, n) = \frac{1}{3} \mathcal{N}(A_1 \mathcal{P} D_4, n, 0) \).

**Proof.** This is identical to the proof of the corresponding lemma in [1]. \( \square \)

To summarize, the definition of \( \mathcal{N}(A_1 x_k, n) \) is (2.2). Lemma 2.9 equates this number to a quantity amenable to topological computational methods. We then introduce another number \( \mathcal{N}(A_1 \mathcal{P} x_k, n, m) \) in Definition 2.12 and relate it to \( \mathcal{N}(A_1 x_k, n) \) in Lemma 2.13. In other words, we do not compute \( \mathcal{N}(A_1 x_k, n) \) directly; we compute it indirectly by first computing \( \mathcal{N}(A_1 \mathcal{P} x_k, n, m) \) and then using Lemma 2.13.

We now give a brief idea of how to compute these numbers. Suppose we want to compute \( \mathcal{N}(A_1 \mathcal{P} x_k, n, m) \). We first find some singularity \( \tilde{x}_l \) for which \( \mathcal{N}(A_1 \mathcal{P} x_l, n, m) \) has been calculated and which contains \( x_k \) in its closure, i.e., we want \( \mathcal{P} x_k \) to be a subset of \( \mathcal{P} \tilde{x}_l \). Usually, \( l = k - 1 \) but it is not necessary. Our next task is to describe the closure of \( \mathcal{P} x_l \) and \( A_1 \circ \mathcal{P} x_l \) explicitly as

\[ \mathcal{P} \tilde{x}_l = \mathcal{P} x_l \cup \mathcal{P} x_k \cup B_1 \quad \text{and} \quad A_1 \circ \mathcal{P} x_l = A_1 \circ \mathcal{P} x_l \cup A_1 \circ (\mathcal{P} x_l - \mathcal{P} x_l) \cup (\Delta B_2) \]  

(2.5)

\[ \begin{align*}
\Delta B_2 &= \{ (\tilde{f}, \tilde{p}_1, l_{\tilde{p}_2}) \in A_1 \circ \mathcal{P} x_l : \tilde{p}_1 = \tilde{p}_2 \}.
\end{align*} \]  

(2.6)

Note that \( A_1 \circ \mathcal{P} x_l = \mathcal{A}_1 \circ \mathcal{P} x_l \). The main content of [1] was to express \( \mathcal{P} \tilde{x}_l \) as in (2.5). The main content of this paper is to compute \( \Delta B_2 \), i.e., expressing \( \mathcal{A}_1 \circ \mathcal{P} x_l \) as in (2.6). Concretely, computing \( \Delta B_2 \) means figuring out what happens to a \( x_k \) singularity, when it collides with an \( A_1 \)-node. As a simple example, when two nodes collide, we get an \( A_3 \)-node (Fig. 1).

This is basically the content of Lemma 6.3 (2). By Definition 2.12 and Theorem 2.3

\[ \mathcal{N}(A_1 \mathcal{P} x_k, n, m) := \langle e(\mathbb{W}^1_{n,m,k}), [\mathcal{A}_1 \circ \mathcal{P} x_k] \rangle = |\pm Q^{-1}(0) \cap A_1 \circ \mathcal{P} x_k|, \]

where
\( Q : D \times \mathbb{P}^2 \times \mathbb{P}T^2 \rightarrow W^\delta_{n,m,k} := \left( \bigoplus_{i=1}^{\delta_d-(n+m+k+\delta)} \pi^*_D \gamma^*_D \right) \oplus \left( \bigoplus_{i=1}^{n} \pi^*_1 \gamma^*_1 \right) \oplus \left( \bigoplus_{i=1}^{m} \pi^*_2 \gamma^*_2 \right) \) (2.7)

is a generic smooth section. In [1], we constructed a section \( \psi_{\mathcal{P} \mathcal{X}_k} \) of an appropriate vector bundle

\[ V_{\mathcal{P} \mathcal{X}_k} \rightarrow \overline{\mathcal{P} \mathcal{X}_k} = \mathcal{P} \mathcal{X}_1 \cup \overline{\mathcal{P} \mathcal{X}_k} \cup \mathcal{B}_1 \]

with the following properties: the section \( \psi_{\mathcal{P} \mathcal{X}_k} : \overline{\mathcal{P} \mathcal{X}_k} \rightarrow V_{\mathcal{P} \mathcal{X}_k} \) does not vanish on \( \mathcal{P} \mathcal{X}_1 \) and it vanishes transversely on \( \mathcal{P} \mathcal{X}_k \). With a similar reasoning one can show that the induced section

\[ \pi^*_2 \psi_{\mathcal{P} \mathcal{X}_k} : \mathcal{A}_1 \circ \mathcal{P} \mathcal{X}_1 \rightarrow \pi^*_2 V_{\mathcal{P} \mathcal{X}_k} \]

vanishes transversely on \( \mathcal{A}_1 \circ \mathcal{P} \mathcal{X}_k \).\(^4\) Here \( \pi_2 \) is the following projection map

\[ \pi_2 : D \times \mathbb{P}^2 \times \mathbb{P}T^2 \rightarrow D \times \mathbb{P}T^2. \]

Since \( \psi_{\mathcal{P} \mathcal{X}_k} \) does not vanish on \( \mathcal{P} \mathcal{X}_1 \), the section \( \pi^*_2 \psi_{\mathcal{P} \mathcal{X}_k} \) does not vanish on \( \mathcal{A}_1 \circ \mathcal{P} \mathcal{X}_1 \). Therefore,

\[ \langle e(\pi^*_2 V_{\mathcal{P} \mathcal{X}_k} \oplus W^1_{n,m,k}), [\overline{\mathcal{A}_1 \circ \mathcal{P} \mathcal{X}_1}] \rangle = \mathcal{N}(\mathcal{A}_1 \mathcal{P} \mathcal{X}_k, n, m) + \mathcal{C}_{\mathcal{A}_1 \circ \mathcal{B}_1}(\pi^*_2 \psi_{\mathcal{P} \mathcal{X}_k} \oplus \mathcal{Q}) + \mathcal{C}_{\Delta \mathcal{B}_2}(\pi^*_2 \psi_{\mathcal{P} \mathcal{X}_k} \oplus \mathcal{Q}) \] (2.8)

where \( \mathcal{C}_{\mathcal{A}_1 \circ \mathcal{B}_1}(\pi^*_2 \psi_{\mathcal{P} \mathcal{X}_k} \oplus \mathcal{Q}) \) and \( \mathcal{C}_{\Delta \mathcal{B}_2}(\pi^*_2 \psi_{\mathcal{P} \mathcal{X}_k} \oplus \mathcal{Q}) \) are the contributions of the section \( \pi^*_2 \psi_{\mathcal{P} \mathcal{X}_k} \oplus \mathcal{Q} \) to the Euler class from the points of \( \mathcal{A}_1 \circ \mathcal{B}_1 \) and \( \Delta \mathcal{B}_2 \) respectively. The number \( \mathcal{C}_{\mathcal{A}_1 \circ \mathcal{B}_1}(\pi^*_2 \psi_{\mathcal{P} \mathcal{X}_k} \oplus \mathcal{Q}) \) was computed in [1]. The main content of this paper is to compute \( \mathcal{C}_{\Delta \mathcal{B}_2}(\pi^*_2 \psi_{\mathcal{P} \mathcal{X}_k} \oplus \mathcal{Q}) \). Once we have computed these numbers, we observe that the left hand side of (2.8) is computable via splitting principle and the fact that \( \mathcal{N}(\mathcal{A}_1 \mathcal{P} \mathcal{X}_1, n, m) \) is known. Therefore, we get a recursive formula for \( \mathcal{N}(\mathcal{A}_1 \mathcal{P} \mathcal{X}_k, n, m) \) in terms of \( \mathcal{N}(\mathcal{A}_1 \mathcal{P} \mathcal{X}_l, n', m') \) and \( \mathcal{N}(\mathcal{P} \mathcal{X}_{k+1}, n, m) \). The main result of [1] was to find an

---

\(^4\) However the bound on \( d \) for which transversality is achieved increases.
explicit formula for $\mathcal{N}(\mathcal{P}\mathcal{X}_{k+1}, n, m)$. Using this and iterations, we get an explicit formula for $\mathcal{N}(A_1\mathcal{P}\mathcal{X}_k, n, m)$. Finally, using Lemma 2.4, we get our desired numbers $\mathcal{N}(A_1\mathcal{X}_k, n)$.

**Example 2.14.** Suppose we wish to compute $\mathcal{N}(A_1A_5, n)$. This can be deduced from the knowledge of $\mathcal{N}(A_1\mathcal{P}A_5, n, m)$. The obvious singularities which have $A_5$-nodes in its closure are $A_4$-nodes. In order to analyze the space $\overline{A_1 \circ \mathcal{P} A_4}$, we infer (cf. Lemma 6.3 (6)) that

$$\overline{A_1 \circ \mathcal{P} A_4} = \overline{A_1 \circ \mathcal{P} A_4} \cup \overline{\mathcal{A}_1 \circ (\mathcal{P} A_4 - \mathcal{P} A_4)} \cup (\Delta \mathcal{P} A_6 \cup \Delta \mathcal{P} \mathcal{D}_7 \cup \Delta \mathcal{P} \mathcal{E}_6).$$

By Lemma 6.1 (9) we conclude that

$$\overline{A_1 \circ \mathcal{P} A_4} = \overline{A_1 \circ \mathcal{P} A_4} \cup \overline{\mathcal{A}_1 \circ (\mathcal{P} A_5 \cup \mathcal{P} \mathcal{D}_5)} \cup (\Delta \mathcal{P} A_6 \cup \Delta \mathcal{P} \mathcal{D}_7 \cup \Delta \mathcal{P} \mathcal{E}_6).$$

The corresponding line bundle $L_{\mathcal{P}A_5} \rightarrow \overline{\mathcal{P} A_4}$ with a section $\psi_{\mathcal{P} A_5}$ that does not vanish on $\mathcal{P} A_4$ and vanishes transversely on $\mathcal{P} A_5$ is defined in Section 4.1. In Section 5, we indicate that

$$\pi_2^* \psi_{\mathcal{P} A_5} : \overline{A_1 \circ \mathcal{P} A_4} \longrightarrow \pi_2^* L_{\mathcal{P} A_5}$$

vanishes on $A_1 \circ \mathcal{P} A_5$ transversely. Let $\mathcal{Q}$ be a generic section of the vector bundle

$$\mathcal{W}_{n,m,5}^l \longrightarrow \mathcal{D} \times \mathbb{P}^2 \times \mathbb{P} \mathbb{P}^2.$$ 

In [1] we show that $\pi_2^* \psi_{\mathcal{P} A_5} \oplus \mathcal{Q}$ vanishes on all points of $A_1 \circ \mathcal{P} \mathcal{D}_5$ with a multiplicity of 2. By Corollary 6.13 and 6.20, $\pi_2^* \psi_{\mathcal{P} A_5} \oplus \mathcal{Q}$ vanishes on all the points of $\Delta \mathcal{P} A_6$ and $\Delta \mathcal{P} \mathcal{E}_6$ with a multiplicity of 2 and 5 respectively. Furthermore, we also show that $\Delta \mathcal{P} \mathcal{D}_7$ is contained inside $\Delta \mathcal{P} \mathcal{D}_7$. Since the dimension of $\Delta \mathcal{P} \mathcal{D}_7$ is one less than the rank of $\pi_2^* L_{\mathcal{P} A_5} \oplus \mathcal{W}_{n,m,5}^l$ and $\mathcal{Q}$ is generic, $\pi_2^* \psi_{\mathcal{P} A_5} \oplus \mathcal{Q}$ does not vanish on $\Delta \mathcal{P} \mathcal{D}_7$. Hence, it does not vanish on $\Delta \mathcal{P} \mathcal{D}_7$. Therefore, we conclude that

$$\langle e(\pi_2^* L_{\mathcal{P} A_5} \oplus \mathcal{W}_{n,m,5}^l), [\overline{A_1 \circ \mathcal{P} A_4}] \rangle = \mathcal{N}(A_1\mathcal{P}A_5, n, m) + 2\mathcal{N}(A_1\mathcal{P}D_5, n, m)$$

$$+ 2\mathcal{N}(\mathcal{P}A_6, n, m) + 5\mathcal{N}(\mathcal{P}E_6, n, m).$$

This gives us a recursive formula for $\mathcal{N}(A_1\mathcal{P}A_5, n, m)$ in terms of $\mathcal{N}(A_1\mathcal{P}A_4, n', m'), \mathcal{N}(\mathcal{P}A_6, n, m), \mathcal{N}(\mathcal{P}D_5, n, m)$ and $\mathcal{N}(\mathcal{P}E_6, n, m)$, which is (3.6) in our algorithm.

**Remark 2.15.** We remind the reader that $\mathcal{N}(\mathcal{P}\mathcal{X}_k, n, m)$ has been defined in [1]. The definition is analogous to the definition of $\mathcal{N}(A_1\mathcal{P}X_k, n, m)$ as given in Definition 2.12 in this paper.

Now we describe the basic organization of our paper. In Section 3 we state the basic identities that are used recursively to obtain the numbers $\mathcal{N}(A_1\mathcal{X}_k, n)$ in our MAIN
THEOREM. In Section 4 we recapitulate all the spaces, vector bundles and sections of vector bundles we encountered in the process of enumerating curves with one singular point. In Section 5 we introduce a few new notation needed for this paper and write down the relevant sections that are transverse to the zero set. The proof of why the sections are transverse to the zero set can be found in [2]. In Section 6 we stratify the space $\bar{X} \circ \mathcal{P}X_k$ as described in (2.6). Along the way we also compute the order to which a certain section vanishes around certain points (i.e., the contribution of the section to the Euler class of a bundle). Finally, using the splitting principal, in Section 7 we compute the Euler class of the relevant bundles and obtain the recursive formula similar to (2.9) above.

3. The basic identities

We now provide the basic identities used to compute the numbers $N(A_1X_k, n)$. Eqs. (3.2)–(3.13) are recursive formulas for $N(A_1\mathcal{P}X_k, n, m)$ in terms of $N(A_1\mathcal{P}X_{k-1}, n', m')$ and $N(\mathcal{P}X_{k+1}, n, m)$. In [1] we had obtained an explicit formula for $N(\mathcal{P}X_{k+1}, n, m)$. Finally, using Lemma 2.4, we get our desired numbers $N(A_1X_k, n)$. We have implemented this algorithm in a Mathematica program to obtain the final answers (cf. Section 3.1). The program is available on our web page

https://www.sites.google.com/site/ritwik371/home.

We prove these identities in Section 7.

First we note that using the ring structure of $H^*(D \times \mathbb{P}^2 \times \mathbb{PT}\mathbb{P}^2; \mathbb{Z})$, it is easy to see that for every singularity type $X_k$ we have

$$N(A_1\mathcal{P}X_k, n, m) = -3N(A_1\mathcal{P}X_k, n + 1, m - 1) - 3N(A_1\mathcal{P}X_k, n + 2, m - 2)$$

$$\forall m \geq 2.$$  \tag{3.1}

We now give recursive formulas for $N(A_1A_1, n)$ and $N(A_1\mathcal{P}X_k, n, m)$:

$$N(A_1A_1, n) = N(A_1, 0) \times N(A_1, n)$$

$$- (N(A_1, n) + dN(A_1, n + 1) + 3N(A_2, n))$$  \tag{3.2}

$$N(A_1\mathcal{P}A_2, n, 0) = 2N(A_1A_1, n) + 2(d - 3)N(A_1A_1, n + 1)$$

$$- 2N(\mathcal{P}A_3, n, 0)$$  \tag{3.3}

$$N(A_1\mathcal{P}A_2, n, 1) = N(A_1A_1, n) + (2d - 9)N(A_1A_1, n + 1)$$

$$+ (d^2 - 9d + 18)N(A_1A_1, n + 2)$$

$$- 2N(\mathcal{P}A_3, n, 1) - 3N(\mathcal{D}_4, n)$$  \tag{3.4}
\[ N(A_1 P A_3, n, m) = N(A_1 P A_2, n, m) + 3N(A_1 P A_2, n, m + 1) + dN(A_1 P A_2, n + 1, m) - 2N(P A_4, n, m) \]

(3.5)

\[ N(A_1 P A_4, n, m) = 2N(A_1 P A_3, n, m) + 2N(A_1 P A_3, n, m + 1) \]

+ \((2d - 6)N(A_1 P A_3, n + 1, m) - 2N(P A_5, n, m)\)

(3.6)

\[ N(A_1 P A_5, n, m) = 3N(A_1 P A_4, n, m) + N(A_1 P A_4, n, m + 1) \]

+ \((3d - 12)N(A_1 P A_4, n + 1, m) - 2N(A_1 P D_5, n, m) - 2N(P A_6, n, m) - 5N(P E_6, n, m)\)

(3.7)

\[ N(A_1 P A_6, n, m) = 4N(A_1 P A_5, n, m) + 0N(P A_1 A_5, n, m + 1) \]

+ \((4d - 18)N(P A_1 A_5, n + 1, m) - 4N(A_1 P D_6, n, m) - 3N(A_1 P E_6, n, m) \]

- \(2N(P A_7, n, m) - 6N(P E_7, n, m)\)

(3.8)

\[ N(A_1 P D_4, n, 0) = N(A_1 P A_3, n, 0) - 2N(A_1 P A_3, n, 1) + (d - 6)N(A_1 P A_3, n + 1, 0) \]

- \(2N(D_5, n)\)

(3.9)

\[ N(A_1 P D_4, n, 1) = N(A_1 D_4, n, 0) + (d - 9)N(A_1 D_4, n + 1, 0) \]

(3.10)

\[ N(A_1 P D_5, n, m) = N(A_1 P D_4, n, m) + N(A_1 P D_4, n, m + 1) \]

+ \((d - 3)N(A_1 P D_4, n + 1, m) - 2N(P D_6, n, m)\)

(3.11)

\[ N(A_1 P D_6, n, m) = N(A_1 P D_5, n, m) + 4N(A_1 P D_5, n, m + 1) + dN(A_1 P D_5, n + 1, m) \]

- \(2N(P D_7, n, m) - N(P E_7, n, m)\)

(3.12)

\[ N(A_1 P E_6, n, m) = N(A_1 P D_5, n, m) - N(A_1 P D_5, n, m + 1) \]

+ \((d - 6)N(A_1 P D_5, n + 1, m) - N(P E_7, n, m)\)

(3.13)

3.1. Explicit formulas

For the convenience of the reader, we explicitly write down the formula for \(N(A_1 x_k, n)\).

\[ N(A_1 A_1, 0) = 3(d - 1)(d - 2)(3d^2 - 3d - 11) \]

\[ N(A_1 A_1, 1) = 9d^3 - 27d^2 - d + 30 \]

\[ N(A_1 A_1, 2) = 3d^2 - 6d - 4 \]

\[ N(A_1 A_2, 0) = 12(d - 3)(3d^3 - 6d^2 - 11d + 18) \]

\[ N(A_1 A_2, 1) = 12(d - 3)(2d^2 - d - 5) \]

\[ N(A_1 A_2, 2) = 6(d - 3)(d + 1) \]

\[ N(A_1 A_3, 0) = 6(d - 3)(25d^3 - 71d^2 - 122d + 280) \]
\[ N(A_1A_3, 1) = 3(25d^3 - 98d^2 - 47d + 316) \]
\[ N(A_1A_3, 2) = 3(5d^2 - 10d - 23) \]
\[ N(A_1A_4, 0) = 60(9d^4 - 60d^3 + 47d^2 + 330d - 506) \]
\[ N(A_1A_4, 1) = 4(54d^3 - 234d^2 - 149d + 1029) \]
\[ N(A_1A_4, 2) = 4(9d^2 - 18d - 56) \]
\[ N(A_1A_5, 0) = 9(210d^4 - 1560d^3 + 1432d^2 + 10391d - 18183) \]
\[ N(A_1A_5, 1) = 9(70d^3 - 330d^2 - 262d + 1867) \]
\[ N(A_1A_5, 2) = 9(10d^2 - 20d - 79) \]
\[ N(A_1A_6, 0) = 21(316d^4 - 2567d^3 + 2647d^2 + 20215d - 39372) \]
\[ N(A_1A_6, 1) = 3(632d^3 - 3199d^2 - 3037d + 22292) \]
\[ N(A_1A_6, 2) = 237d^2 - 474d - 2276 \]
\[ N(A_1D_4, 0) = 3(d - 2)(15d^3 - 60d^2 - 65d + 314) \]
\[ N(A_1D_4, 1) = 2(9d^3 - 36d^2 - 25d + 138) \]
\[ N(A_1D_4, 2) = 3d^2 - 6d - 17 \]
\[ N(A_1D_5, 0) = 12(21d^4 - 141d^3 + 109d^2 + 814d - 1308) \]
\[ N(A_1D_5, 1) = 6(14d^3 - 61d^2 - 48d + 293) \]
\[ N(A_1D_5, 2) = 12(d^2 - 2d - 7) \]
\[ N(A_1D_6, 0) = 3(224d^4 - 1666d^3 + 1484d^2 + 11598d - 20793) \]
\[ N(A_1D_6, 1) = 3(64d^3 - 302d^2 - 281d + 1827) \]
\[ N(A_1D_6, 2) = 3(8d^2 - 16d - 69) \]
\[ N(A_1E_6, 0) = 9(d - 4)(28d^3 - 91d^2 - 177d + 567) \]
\[ N(A_1E_6, 1) = 9(d - 4)(8d^2 - 5d - 51) \]
\[ N(A_1E_6, 2) = 9(d - 4)(d + 2) \]

4. Review of definitions and notations for one singular point

We recall a few definitions and notation from [1] and [2] so that our paper is self-contained.
4.1. The vector bundles involved

The first three of the vector bundles we will encounter, the tautological line bundles, have been defined in Notations 2.8 and 2.11. Let \( \pi : \mathcal{D} \times \mathbb{P}T^2 \to \mathcal{D} \times \mathbb{P}^2 \) be the projection map. We define the following bundles over \( \mathcal{D} \times \mathbb{P}^2 \) and \( \mathcal{D} \times \mathbb{P}T^2 \).

\[
\begin{align*}
\mathcal{L}_{A_0} & := \gamma^*_{\mathcal{D}} \otimes \gamma^*_{\mathbb{P}^2} \to \mathcal{D} \times \mathbb{P}^2 \\
\mathcal{V}_{A_1} & := \gamma^*_{\mathcal{D}} \otimes \gamma^*_{\mathbb{P}^2} \otimes T^* \mathbb{P}^2 \to \mathcal{D} \times \mathbb{P}^2 \\
\mathcal{V}_{P_{A_2}} & := \hat{\gamma}^* \otimes \gamma^*_{\mathcal{D}} \otimes \gamma^*_{\mathbb{P}^2} \otimes \pi^* T^* \mathbb{P}^2 \to \mathcal{D} \times \mathbb{P}T^2 \\
\mathcal{L}_{P_{D_4}} & := (T^* \mathbb{P}^2 / \hat{\gamma})^* \otimes \gamma^*_{\mathcal{D}} \otimes \gamma^*_{\mathbb{P}^2} \to \mathcal{D} \times \mathbb{P}T^2 \\
\mathcal{L}_{P_{D_5}} & := \hat{\gamma}^* \otimes (T^* \mathbb{P}^2 / \hat{\gamma})^* \otimes \gamma^*_{\mathcal{D}} \otimes \gamma^*_{\mathbb{P}^2} \to \mathcal{D} \times \mathbb{P}T^2 \\
\mathcal{L}_{P_{E_6}} & := \hat{\gamma}^* \otimes (T^* \mathbb{P}^2 / \hat{\gamma})^* \otimes \gamma^*_{\mathcal{D}} \otimes \gamma^*_{\mathbb{P}^2} \to \mathcal{D} \times \mathbb{P}T^2 \\
\mathcal{L}_{P_{E_7}} & := \hat{\gamma}^* \otimes \gamma^*_{\mathcal{D}} \otimes \gamma^*_{\mathbb{P}^2} \to \mathcal{D} \times \mathbb{P}T^2
\end{align*}
\]

where \( \epsilon_6 = 0, \epsilon_7 = 1 \) and \( \epsilon_8 = 3 \).

4.2. Sections of vector bundles

We assume that the reader is familiar with the notion of vertical derivatives; the definition can be found in [2]. Let \( f : \mathbb{P}^2 \to \gamma^*_{\mathbb{P}^2} \) be a section and \( \tilde{p} \in \mathbb{P}^2 \). We can think of \( p \) as a non-zero vector in \( \gamma_{\mathbb{P}^2} \) and \( p^{\otimes d} \) a non-zero vector in \( \gamma^{\otimes d}_{\mathbb{P}^2} \). The vertical derivative \( \nabla f \big|_{\tilde{p}} \) acts on a vector in \( \gamma^*_{\mathbb{P}^2} \big|_{\tilde{p}} \) and produces an element of \( T^*_{\tilde{p}} \mathbb{P}^2 \). Let us denote this quantity as \( \nabla f \big|_{\tilde{p}} \), i.e.,

\[
\nabla f \big|_{\tilde{p}} := \{ \nabla f \big|_{\tilde{p}} \} (p^{\otimes d}) \in T^*_{\tilde{p}} \mathbb{P}^2.
\]

Notice that \( \nabla f \big|_{\tilde{p}} \) is an element of the fibre of \( T^* \mathbb{P}^2 \otimes \gamma^*_{\mathbb{P}^2} \) at \( \tilde{p} \) while \( \nabla f \big|_{\tilde{p}} \) is an element of \( T^*_{\tilde{p}} \mathbb{P}^2 \).

Now observe that \( \pi^* \mathbb{P}^2 \cong \hat{\gamma} \oplus \pi^* T^* \mathbb{P}^2 / \hat{\gamma} \to \mathbb{P}T^2 \mathbb{P}^2 \), where \( \pi : \mathbb{P}T^2 \mathbb{P}^2 \to \mathbb{P}^2 \) is the projection map. Let us denote a vector in \( \hat{\gamma} \) by \( v \) and a vector in \( \pi^* T^* \mathbb{P}^2 / \hat{\gamma} \) by \( \hat{w} \). Given \( \tilde{f} \in \mathcal{D} \) and \( \tilde{p} \in \mathbb{P}^2 \), let

\[\text{We will make the abuse of notation of usually omitting the pullback maps } \pi^*_{\tilde{p}} \text{ and } \pi^*_{\tilde{p}}. \text{ Our intended meaning should be clear when we say, for instance, } \gamma^*_{\mathcal{D}} \to \mathcal{D} \times \mathbb{P}^2. \text{ However, we will not omit to write the pullback via } \pi^*. \]

\[\text{Recall that } p \text{ is an element of } \mathbb{C}^3 - 0 \text{ while } \tilde{p} \text{ is the corresponding equivalence class in } \mathbb{P}^2. \]
\[ f_{ij} := \nabla^{i+j} f|_p(v, \ldots, v, w, \ldots w) \]

Note that \( f_{ij} \) is a number. In general \( f_{ij} \) is not well defined on the whole space; it depends on the trivialization of the bundle. Moreover it is also not well defined on the quotient space. Since our sections are not defined on the whole space, we will use the notation \( s : M \rightarrow V \) to indicate that \( s \) is defined only on a subspace of \( M \). With this terminology, we now explicitly define the sections that we will encounter in this paper.

\[
\psi_{A_0} : D \times \mathbb{P}^2 \rightarrow L_{A_0}, \quad \{\psi_{A_0}(\tilde{f}, \tilde{p})\}(f \otimes p^{\otimes d}) := f(p)
\]

\[
\psi_{A_1} : D \times \mathbb{P}^2 \rightarrow \mathcal{V}_{A_1}, \quad \{\psi_{A_1}(\tilde{f}, \tilde{p})\}(f \otimes p^{\otimes d}) := \nabla f|_p
\]

We also have

\[
\psi_{P_{A_2}} : D \times \mathbb{P}T^2 \rightarrow \mathcal{V}_{P_{A_2}}, \quad \{\psi_{P_{A_2}}(\tilde{f}, l_p)\}(f \otimes p^{\otimes d} \otimes v) := \nabla^2 f|_p(v, \cdot)
\]

\[
\psi_{P_{D_4}} : D \times \mathbb{P}T^2 \rightarrow \mathbb{L}_{P_{D_4}}, \quad \{\psi_{P_{D_4}}(\tilde{f}, l_p)\}(f \otimes p^{\otimes d} \otimes w^{\otimes 2}) := f_{02}
\]

\[
\psi_{P_{D_5}} : D \times \mathbb{P}T^2 \rightarrow \mathbb{L}_{P_{D_5}}, \quad \{\psi_{P_{D_5}}(\tilde{f}, l_p)\}(f \otimes p^{\otimes d} \otimes v^{\otimes 2} \otimes w) := f_{21}
\]

\[
\psi_{P_{E_6}} : D \times \mathbb{P}T^2 \rightarrow \mathbb{L}_{P_{E_6}}, \quad \{\psi_{P_{E_6}}(\tilde{f}, l_p)\}(f \otimes p^{\otimes d} \otimes v \otimes w^{\otimes 2}) := f_{12}
\]

\[
\psi_{P_{E_7}} : D \times \mathbb{P}T^2 \rightarrow \mathbb{L}_{P_{E_7}}, \quad \{\psi_{P_{E_7}}(\tilde{f}, l_p)\}(f \otimes p^{\otimes d} \otimes v^{\otimes 4}) := f_{40}
\]

Next, let us define the following numbers \( A_{3}^k \) (for \( 3 \leq k \leq 7 \)) and \( D_{k}^f \) (for \( 6 \leq k \leq 8 \)):

\[
A_{3}^3 := f_{30}, \quad A_{3}^4 := f_{40} - \frac{3f_{21}^2}{f_{02}}, \quad A_{3}^5 := f_{50} - \frac{10f_{21}f_{31}}{f_{02}} + \frac{15f_{12}f_{21}^2}{f_{02}^2},
\]

\[
A_{6}^6 := f_{60} - \frac{15f_{21}f_{41}}{f_{02}} - \frac{10f_{21}^2}{f_{02}} - \frac{60f_{12}f_{21}f_{31}}{f_{02}^2} + \frac{45f_{21}^2f_{22}}{f_{02}^2} - \frac{15f_{03}f_{31}^3}{f_{02}^3} - \frac{90f_{12}^2f_{21}^2}{f_{02}^3},
\]

\[
A_{7}^7 := f_{70} - \frac{21f_{21}f_{51}}{f_{02}} - \frac{35f_{31}f_{41}}{f_{02}} + \frac{105f_{12}f_{21}f_{41}}{f_{02}^2} + \frac{105f_{21}^2f_{32}}{f_{02}^2} + \frac{70f_{12}f_{31}^3}{f_{02}^3} + \frac{210f_{21}f_{22}f_{31}}{f_{02}^2} + \frac{105f_{03}f_{21}^3}{f_{02}^3} + \frac{420f_{21}^2f_{21}f_{31}}{f_{02}^3} + \frac{630f_{12}f_{21}f_{41}}{f_{02}^3} + \frac{105f_{13}f_{21}^3}{f_{02}^3} + \frac{315f_{03}f_{12}f_{21}^3}{f_{02}^3} + \frac{630f_{12}f_{21}^2}{f_{02}^3} \quad (4.1)
\]

and

\[
D_{6}^6 := f_{40}, \quad D_{7}^7 := f_{50} - \frac{5f_{31}^2}{3f_{12}},
\]

\[
D_{8}^8 := f_{60} + \frac{5f_{03}f_{31}f_{50}}{3f_{12}^2} - \frac{5f_{31}f_{41}}{f_{12}} - \frac{10f_{03}f_{31}^3}{3f_{12}^3} + \frac{5f_{22}f_{31}}{f_{12}^2} \quad (4.2)
\]
We now define the sections $\Psi_{PA_k} : D \times \mathbb{P}T^2 \to L_{PA_k}$ (for $3 \leq k \leq 7$) and $\Psi_{PD_k} : D \times \mathbb{P}T^2 \to L_{PD_k}$ (for $6 \leq k \leq 8$) given by

$$\{\Psi_{PA_k}(\tilde{f}, l_p)\} \left( f^\otimes (k-2) \otimes p^\otimes d(k-2) \otimes t^\otimes k \otimes w^\otimes (2k-6) \right) := f_{ij}^{k-3} A_k^f,$$

$$\{\Psi_{PD_k}(\tilde{f}, l_p)\} \left( f^\otimes (1+\epsilon_k) \otimes p^\otimes d(1+\epsilon_k) \otimes v^\otimes (k-2+\epsilon_k) \otimes w^\otimes (2\epsilon_k) \right) := f_{ij}^{k} D_k^f,$$

where $\epsilon_6 = 0$, $\epsilon_7 = 1$ and $\epsilon_8 = 3$. In [2], we state and prove on which subspaces these sections are well defined.

4.3. The spaces involved

We begin by explaining a terminology. If $l_p \in \mathbb{P}T^2 \mathbb{P}^2$, then we say that $v \in l_p$ if $v$ is a tangent vector in $T_p \mathbb{P}^2$ and lies over the fibre of $l_p$. We now define the spaces that we will encounter.

$$X_k := \{(\tilde{f}, p) \in D \times \mathbb{P}^2 : f \text{ has a singularity of type } X_k \text{ at } p\}$$

$$\tilde{X}_k := \{(\tilde{f}, l_p) \in D \times \mathbb{P}T^2 : f \text{ has a singularity of type } X_k \text{ at } p\}$$

$$\pi^{-1}(X_k)$$

if $k > 1$ $PA_k := \{(\tilde{f}, l_p) \in D \times \mathbb{P}T^2 : f \text{ has a singularity of type } A_k \text{ at } p,$

$$\nabla^2 f|_p (v, \cdot) = 0 \text{ if } v \in l_p\}$$

$$PD_4 := \{(\tilde{f}, l_p) \in D \times \mathbb{P}T^2 : f \text{ has a singularity of type } D_4 \text{ at } p,$

$$\nabla^3 f|_p (v, v, v) = 0 \text{ if } v \in l_p\}$$

if $k > 4$ $PD_k := \{(\tilde{f}, l_p) \in D \times \mathbb{P}T^2 : f \text{ has a singularity of type } D_k \text{ at } p,$

$$\nabla^3 f|_p (v, v, \cdot) = 0 \text{ if } v \in l_p\}$$

if $k = 6, 7$ or $8$ $PE_k := \{(\tilde{f}, l_p) \in D \times \mathbb{P}T^2 : f \text{ has a singularity of type } E_k \text{ at } p,$

$$\nabla^3 f|_p (v, v, \cdot) = 0 \text{ if } v \in l_p\}$$

if $k > 4$ $PD_k^\epsilon := \{(\tilde{f}, l_p) \in D \times \mathbb{P}T^2 : f \text{ has a singularity of type } D_k \text{ at } p,$

$$\nabla^3 f|_p (v, v, v) = 0, \nabla^3 f|_p (v, v, w) \neq 0$$

if $v \in l_p$ and $w \in (T_p \mathbb{P}^2)/l_p\}$$

We also need the definitions for a few other spaces which will make our computations convenient.

$$\tilde{A}_1^# := \{(\tilde{f}, l_p) \in D \times \mathbb{P}T^2 : f(p) = 0, \nabla f|_p = 0, \nabla^2 f|_p (v, \cdot) \neq 0, \forall v \neq 0 \in l_p\}$$

$$\tilde{D}_4^# := \{(\tilde{f}, l_p) \in D \times \mathbb{P}T^2 : f(p) = 0, \nabla f|_p = 0, \nabla^2 f|_p = 0, \nabla^3 f|_p (v, v, v) \neq 0, \forall v \neq 0 \in l_p\}$$
\[
\hat{D}_k^\# := \{(\tilde{f}, l_{\tilde{p}}) \in D \times \mathbb{P}TP^2 : f \text{ has a } D_k \text{ singularity at } \tilde{p}, \nabla^3 f|_{\tilde{p}}(v, v, v) \neq 0, \\
\quad \forall v \neq 0 \in l_{\tilde{p}}, \ k \geq 4\}
\]
\[
\hat{X}_8^\# := \{(\tilde{f}, l_{\tilde{p}}) \in D \times \mathbb{P}TP^2 : f(p) = 0, \nabla f|_p = 0, \nabla^2 f|_p \equiv 0, \nabla^3 f|_p = 0, \\
\quad \nabla^4 f|_p(v, v, v, v) \neq 0 \forall v \neq 0 \in l_{\tilde{p}}\}.
\]

5. Transversality

In this section we list down all the relevant bundle sections that are transverse to the zero set. Let us define the following projection maps:

\[
\begin{align*}
\pi_1 & : D \times \mathbb{P}^2 \times \mathbb{P}^2 \longrightarrow D \times \mathbb{P}^2, \\
\pi_2 & : D \times \mathbb{P}^2 \times \mathbb{P}^2 \longrightarrow D \times \mathbb{P}^2, \\
\pi_1 & : D \times \mathbb{P}^2 \times \mathbb{PTP}^2 \longrightarrow D \times \mathbb{P}^2, \\
\pi_2 & : D \times \mathbb{P}^2 \times \mathbb{PTP}^2 \longrightarrow D \times \mathbb{PTP}^2.
\end{align*}
\]

Hence, given a vector bundle over \(D \times \mathbb{P}^2\) or \(D \times \mathbb{PTP}^2\) we obtain a bundle over \(D \times \mathbb{P}^2 \times \mathbb{P}^2\) and \(D \times \mathbb{P}^2 \times \mathbb{PTP}^2\) respectively, via the pullback maps.

A section of a bundle over \(D \times \mathbb{P}^2\) or \(D \times \mathbb{PTP}^2\) induces a section over the corresponding bundle over \(D \times \mathbb{P}^2 \times \mathbb{P}^2\) and \(D \times \mathbb{P}^2 \times \mathbb{PTP}^2\) respectively, via the pullback maps.

**Remark 5.1.** To describe bundles over \(D \times \mathbb{P}^2\) or \(D \times \mathbb{PTP}^2\), we follow the abuse of notation of omitting pullback maps. However, to describe bundles over \(D \times \mathbb{P}^2 \times \mathbb{P}^2\) or \(D \times \mathbb{P}^2 \times \mathbb{PTP}^2\) we will write the pullback maps.

**Lemma 5.2.** Let \(\pi : E \rightarrow M\) be a fibre bundle with compact fibers. Let \(X \subseteq E\) and \(Y \subseteq M\). Then

\[
\begin{align*}
\pi(X) &= \overline{\pi(X)} \quad (5.1) \\
\pi^{-1}(Y) &= \overline{\pi^{-1}(Y)} \quad (5.2)
\end{align*}
\]

**Proof.** Since the fibers are compact, \(\pi\) is a closed map (Tube Lemma). Combined with the fact that \(\pi\) is continuous (5.1) follows. Secondly, equation (5.2) holds for the trivial bundle, hence it also for an arbitrary fibre bundle, since this is a local statement. \(\square\)

**Proposition 5.3.** The sections of the vector bundles

\[
\pi_2^*\psi_{A_0} : \mathcal{A}_1 \times \mathbb{P}^2 - \Delta \mathcal{A}_1 \longrightarrow \pi_2^*\mathcal{L}_{A_0}, \quad \pi_2^*\psi_{A_1} : \pi_2^*\psi_{A_0}^{-1}(0) \longrightarrow \pi_2^*\mathcal{V}_{A_1}
\]

are transverse to the zero set, provided \(d \geq 3\).
Proposition 5.4. The section of the vector bundle $\pi^*_2\Psi_{\mathcal{P}A_2} : \overline{\mathcal{A}_1} \rightarrow \pi^*_2\mathcal{V}_{\mathcal{P}A_2}$ is transverse to the zero set, provided $d \geq 4$.

Proposition 5.5. The sections of the vector bundles

\[
\begin{align*}
\pi^*_2\Psi_{\mathcal{P}A_3} : \overline{\mathcal{A}_1} \otimes \overline{\mathcal{A}_2} & \rightarrow \pi^*_2\mathcal{L}_{\mathcal{P}A_3}, \\
\pi^*_2\Psi_{\mathcal{P}D_4} : \pi^*_2\Psi^{-1}_{\mathcal{P}A_3}(0) & \rightarrow \pi^*_2\mathcal{L}_{\mathcal{P}D_4}, \\
\pi^*_2\Psi_{\mathcal{P}D_5} : \pi^*_2\Psi^{-1}_{\mathcal{P}D_4}(0) & \rightarrow \pi^*_2\mathcal{L}_{\mathcal{P}D_5}
\end{align*}
\]

are transverse to the zero set provided $d \geq 5$.

Proposition 5.6. If $i \geq 4$, then the sections of the vector bundles

\[
\begin{align*}
\pi^*_2\Psi_{\mathcal{P}A_3} : \overline{\mathcal{A}_1} \otimes \overline{\mathcal{A}_2} & \rightarrow \pi^*_2\mathcal{L}_{\mathcal{P}A_3}, \\
\pi^*_2\Psi_{\mathcal{P}A_4} : \pi^*_2\Psi^{-1}_{\mathcal{P}A_3}(0) - \pi^*_2\Psi^{-1}_{\mathcal{P}D_4}(0) & \rightarrow \pi^*_2\mathcal{L}_{\mathcal{P}A_4}, \ldots, \\
\pi^*_2\Psi_{\mathcal{P}A_i} : \pi^*_2\Psi^{-1}_{\mathcal{P}A_{i-1}}(0) - \pi^*_2\Psi^{-1}_{\mathcal{P}D_4}(0) & \rightarrow \pi^*_2\mathcal{L}_{\mathcal{P}A_i}
\end{align*}
\]

are transverse to the zero set provided $d \geq i + 2$.

Proposition 5.7. If $i \geq 6$, then the sections of the vector bundles

\[
\begin{align*}
\pi^*_2\Psi_{\mathcal{P}D_6} : \overline{\mathcal{A}_1} \otimes \overline{\mathcal{D}_5} & \rightarrow \pi^*_2\mathcal{L}_{\mathcal{P}D_6}, \\
\pi^*_2\Psi_{\mathcal{P}D_7} : \pi^*_2\Psi^{-1}_{\mathcal{P}D_6}(0) - \pi^*_2\Psi^{-1}_{\mathcal{P}E_6}(0) & \rightarrow \pi^*_2\mathcal{L}_{\mathcal{P}D_7}, \ldots, \\
\pi^*_2\Psi_{\mathcal{P}D_i} : \pi^*_2\Psi^{-1}_{\mathcal{P}D_{i-1}}(0) - \pi^*_2\Psi^{-1}_{\mathcal{P}E_6}(0) & \rightarrow \pi^*_2\mathcal{L}_{\mathcal{P}D_i}
\end{align*}
\]

are transverse to the zero set provided $d \geq i + 2$.

Proposition 5.8. The section of the vector bundle $\pi^*_2\Psi_{\mathcal{P}E_6} : \overline{\mathcal{A}_1} \otimes \overline{\mathcal{D}_5} \rightarrow \pi^*_2\mathcal{L}_{\mathcal{P}E_6}$ is transverse to the zero set provided $d \geq 5$.

Proposition 5.9. The section of the vector bundle $\pi^*_2\Psi_{\mathcal{P}A_3} : \overline{\mathcal{A}_1} \otimes \overline{\mathcal{D}_4} \rightarrow \pi^*_2\mathcal{L}_{\mathcal{P}A_3}$ is transverse to the zero set provided $d \geq 5$.

Proof. We have omitted the proofs here; they can be found in [2].

6. Closure and Euler class contribution

In this section we compute closure of a singularity with one $A_1$-node. Along the way we also compute how much a certain section contributes to the Euler class of a bundle. But first, let us recapitulate what we know about the closure of one singular point which was proved in [1].
Lemma 6.1. Let $X_k$ be a singularity of type $A_k, D_k, E_k$ or $X_8$. Then the closures are given by:

1. $\overline{A_0} = A_0 \cup \overline{A_1}$ if $d \geq 3$.
2. $\overline{A_1} = \overline{A_1^\#} = \overline{A_1^\#} \cup \overline{PA_2}$ if $d \geq 3$.
3. $\overline{D_4^\#} = \overline{D_4^\#} \cup \overline{PD_4}$ if $d \geq 3$.
4. $\overline{PD_4} = \overline{PD_4} \cup \overline{PD_5} \cup \overline{PD_6^\#}$ if $d \geq 4$.
5. $\overline{PE_0} = \overline{PE_0} \cup \overline{PE_7} \cup \overline{X_8^\#}$ if $d \geq 4$.
6. $\overline{PD_5} = \overline{PD_5} \cup \overline{PD_6} \cup \overline{PD_6^\#}$ if $d \geq 4$.
7. $\overline{PD_6} = \overline{PD_6} \cup \overline{PD_7} \cup \overline{PE_7}$ if $d \geq 5$.
8. $\overline{PA_2} = \overline{PA_2} \cup \overline{PA_3} \cup \overline{D_4^\#}$ if $d \geq 4$.
9. $\overline{PA_3} = \overline{PA_3} \cup \overline{PA_4} \cup \overline{PD_4}$ if $d \geq 5$.
10. $\overline{PA_4} = \overline{PA_4} \cup \overline{PA_5} \cup \overline{PD_5}$ if $d \geq 6$.
11. $\overline{PA_5} = \overline{PA_5} \cup \overline{PA_6} \cup \overline{PD_6} \cup \overline{PE_6}$ if $d \geq 7$.
12. $\overline{PA_6} = \overline{PA_6} \cup \overline{PA_7} \cup \overline{PD_7} \cup \overline{PE_7} \cup \overline{X_8^\#}$ if $d \geq 8$.

Let us now state a few facts about the closure of one singular point that will be required in this paper. These facts were not explicitly stated in [1] because it was not needed in that paper.

Lemma 6.2. We have the following equality (or inclusion) of sets

1. $\overline{A_1} = A_1 \cup \overline{A_2}$ if $d \geq 2$.
2. $\overline{D_4^\#} = \overline{D_4}$ if $d \geq 3$.
3. $\overline{D_k^\#} = \overline{D_k}$ if $k \geq 4$ and $d \geq 3$.
4. $\{(\tilde{f}, l_p) \in \overline{D_5} : \Psi_{PA_3}(\tilde{f}, l_p) = 0\} = \overline{PD_5} \cup \overline{PD_6^\#}$ if $d \geq 3$.
5. $\{(\tilde{f}, l_p) \in \overline{PA_3} : \Psi_{PD_4}(\tilde{f}, l_p) \neq 0\} = \overline{PA_3} \cup \overline{PA_4} \cup \{(\tilde{f}, l_p) \in \overline{PA_{k+2}} : \Psi_{PD_4}(\tilde{f}, l_p) \neq 0\}$, if $k \geq 3$ and $d \geq k + 1$.
6. $\{(\tilde{f}, l_p) \in \overline{PD_6} : \Psi_{PE_6}(\tilde{f}, l_p) \neq 0\} = \overline{PD_6} \cup \overline{PD_{k+1}} \cup \{(\tilde{f}, l_p) \in \overline{PD_{k+2}} : \Psi_{PE_6}(\tilde{f}, l_p) \neq 0\}$, if $k \geq 6$ and $d \geq k - 1$.
7. $\overline{PE_6} \subset \overline{PD_6^\#}$, if $d \geq 3$.

The proofs are straightforward; the details can be found in [2].

Before stating the main results of this section, let us define three more spaces which will be required while formulating some of the lemmas:

\[
\Delta PD_5^s := \{(\tilde{f}, \tilde{p}, l_p) \in \overline{A_1} \circ \overline{PA_1} : \pi_2^* \Psi_{PD_4}(\tilde{f}, \tilde{p}, l_p) = 0, \pi_2^* \Psi_{PE_0}(\tilde{f}, \tilde{p}, l_p) \neq 0\},
\]

\[
\Delta PD_6^s := \{(\tilde{f}, \tilde{p}, l_p) \in \overline{A_1} \circ \overline{PA_5} : \pi_2^* \Psi_{PD_4}(\tilde{f}, \tilde{p}, l_p) = 0, \pi_2^* \Psi_{PE_6}(\tilde{f}, \tilde{p}, l_p) \neq 0\},
\]

\[
\Delta PD_6^{s*} := \{(\tilde{f}, \tilde{p}, l_p) \in \overline{A_1} \circ \overline{PD_4} : \pi_2^* \Psi_{PD_6}(\tilde{f}, \tilde{p}, l_p) \neq 0\}.
\]
We are now ready to state the main lemma.

**Lemma 6.3.** Let $\mathcal{X}_k$ be a singularity of type $A_k$, $D_k$, $E_k$. Then their closures with one $A_1$-node are given by:

1. $\overline{A_1 \circ (D \times \mathbb{P}^2)} = \overline{A_1 \circ (D \times \mathbb{P}^2)} \cup \Delta A_1$ if $d \geq 1$.
2. $\overline{A_1 \circ A_1^\# = A_1 \circ A_1^\#} \cup \overline{A_1 \circ (A_1^\# - A_1^\#)} \cup \Delta A_3$ if $d \geq 3$.
3. $\overline{A_1 \circ PA_2} = \overline{A_1 \circ PA_2} \cup \overline{A_1 \circ (PA_2 - PA_2)} \cup (\Delta PA_4 \cup \Delta D_5^{PA_4})$ if $d \geq 4$.
4. $\overline{A_1 \circ PA_3} = \overline{A_1 \circ PA_3} \cup \overline{A_1 \circ (PA_3 - PA_3)} \cup (\Delta PA_5 \cup \Delta D_6^{PA_5})$ if $d \geq 5$.
5. $\overline{A_1 \circ PA_4} = \overline{A_1 \circ PA_4} \cup \overline{A_1 \circ (PA_4 - PA_4)} \cup (\Delta PA_6 \cup \Delta D_7^{PA_6})$ if $d \geq 6$.
6. $\overline{A_1 \circ PA_5} = \overline{A_1 \circ PA_5} \cup \overline{A_1 \circ (PA_5 - PA_5)} \cup (\Delta PA_7 \cup \Delta D_8^{PA_7})$ if $d \geq 7$.
7. $\overline{A_1 \circ PD_4} = \overline{A_1 \circ PD_4} \cup \overline{A_1 \circ (PD_4 - PD_4)} \cup (\Delta PD_5^{PA_4} \cup \Delta PD_6)$ if $d \geq 4$.
8. $\overline{A_1 \circ PD_5} = \overline{A_1 \circ PD_5} \cup \overline{A_1 \circ (PD_5 - PD_5)} \cup (\Delta PD_7 \cup \Delta PD_8)$ if $d \geq 5$.
9. $\overline{A_1 \circ PD_6} = \overline{A_1 \circ PD_6} \cup \overline{A_1 \circ (PD_6 - PD_6)} \cup (\Delta PD_7 \cup \Delta PD_8)$ if $d \geq 5$.

**Remark 6.4.** Although the statement of Lemma 6.3 (1) is trivial we give an elaborate proof for two reasons. Firstly, along the way, we prove a few other statements that will be required later. Secondly, as a corollary we also compute the contribution of certain sections to the Euler class of relevant bundles.

**Proof of Lemma 6.3 (1).** It suffices to show that

\[
\{ (\tilde{f}, \tilde{p}, \tilde{p}) \in \overline{A_1 \circ (D \times \mathbb{P}^2)} \} = \Delta A_1.
\]  

(6.2)

Clearly the lhs\(^7\) of (6.2) is a subset of its rhs. To show the converse we will prove the following two claims simultaneously:

\[
\overline{A_1 \circ (D \times \mathbb{P}^2)} \supset \Delta (A_1 \cup A_2),
\]

(6.3)

\[
\overline{A_1 \circ A_1} \cap \Delta (A_1 \cup A_2) = \emptyset.
\]

(6.4)

Since $\overline{A_1 \circ (D \times \mathbb{P}^2)}$ is a closed set, (6.3) implies that the rhs of (6.2) is a subset of its lhs.\(^8\) Moreover, (6.4) is not required at all for the proof of Lemma 6.3 (1). However, these statements will be required in the proofs of Lemma 6.3 (2). Furthermore, in the process of proving (6.3) and (6.4), we will also be computing the contribution of certain sections to the Euler class of relevant bundles as a corollary.

**Claim 6.5.** Let $(\tilde{f}, \tilde{p}, \tilde{p}) \in \Delta (A_1 \cup A_2)$. Then there exist solutions

\[
(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), \tilde{p}(t_1)) \in D \times \mathbb{P}^2 \times \mathbb{P}^2
\]

\(^7\) We shall use lhs to denote left hand side and rhs to denote right hand side of an equation.

\(^8\) In fact the full strength of (6.3) is not really needed; we simply need that $\overline{A_1 \circ (D \times \mathbb{P}^2)} \supset \Delta A_1$. 
sufficiently close to \((\hat{f}, \hat{p}, \hat{p})\) to the set of equations
\[
\pi_1^* \psi_{A_0} (\hat{f}(t_1, t_2), \hat{p}(t_1, t_2), \hat{p}(t_1)) = 0,
\]
\[
\pi_1^* \psi_{A_1} (\hat{f}(t_1, t_2), \hat{p}(t_1, t_2), \hat{p}(t_1)) = 0, \quad \hat{p}(t_1, t_2) \neq \hat{p}(t_1). \tag{6.5}
\]
Furthermore, any such solution sufficiently close to \((\hat{f}, \hat{p}, \hat{p})\) satisfies
\[
(\pi_2^* \psi_{A_0} (\hat{f}(t_1, t_2), \hat{p}(t_1, t_2), \hat{p}(t_1)), \pi_2^* \psi_{A_1} (\hat{f}(t_1, t_2), \hat{p}(t_1, t_2), \hat{p}(t_1))) \neq (0, 0). \tag{6.6}
\]
In particular, \((\hat{f}(t_1, t_2), \hat{p}(t_1, t_2), \hat{p}(t_1))\) does not lie in \(\overline{A_1 \circ A_1}\).

It is easy to see that Claim 6.5 proves statements (6.3) and (6.4) simultaneously.

**Proof.** Choose homogeneous coordinates \([X : Y : Z]\) so that \(\hat{p} = [0 : 0 : 1]\) and let \(U_{\hat{p}}\) be a sufficiently small neighbourhood of \(\hat{p}\) inside \(\mathbb{P}^2\). Denote \(\pi_x, \pi_y : U_{\hat{p}} \to \mathbb{C}\) to be the projection maps given by
\[
\pi_x ([X : Y : Z]) := X/Z \quad \text{and} \quad \pi_y ([X : Y : Z]) := Y/Z,
\]
and \(v, w : U_{\hat{p}} \to T\mathbb{P}^2\) the vector fields dual to the one forms \(d\pi_x\) and \(d\pi_y\) respectively. Let \((\hat{f}(t_1, t_2), \hat{p}(t_1)) \in D \times \mathbb{P}^2\) be an arbitrary point that is close to \((\hat{f}, \hat{p})\) and let \(\hat{p}(t_1, t_2)\) be a point in \(\mathbb{P}^2\) that is close to \(\hat{p}(t_1)\). Let
\[
\hat{p}(t_1) := [x_{t_1} : y_{t_1} : 1] \in \mathbb{P}^2, \quad p(t_1) := (x_{t_1}, y_{t_1}, 1) \in \mathbb{C}^3,
\]
\[
\hat{p}(t_1, t_2) := [x_{t_1} + x_{t_2} : y_{t_1} + y_{t_2} : 1] \in \mathbb{P}^2, \quad p(t_1, t_2) := (x_{t_1} + x_{t_2}, y_{t_1} + y_{t_2}, 1) \in \mathbb{C}^3,
\]
\[
\hat{f}(t_1, t_2) \in \mathbb{P}^{d_4}, \quad f(t_1, t_2) \in \mathbb{C}^{d_4+1}.
\]
Define the following numbers:
\[
f_{ij}(t_1, t_2) := \{\nabla^{i+j} f(t_1, t_2)|_{\hat{p}(t_1)(v, \ldots, v, w, \ldots, w)}\}(p(t_1)^{\otimes d}),
\]
\[
F := f_{00}(t_1, t_2) + f_{10}(t_1, t_2)x_{t_2} + f_{01}(t_1, t_2)y_{t_2} + \sum_{i+j=2} \frac{f_{ij}(t_1, t_2)}{i!j!} x_{t_2}^i y_{t_2}^j + \cdots,
\]
\[
F_{x_{t_2}} := f_{10}(t_1, t_2) + f_{20}(t_1, t_2)x_{t_2} + f_{11}(t_1, t_2)y_{t_2} + \cdots,
\]
\[
F_{y_{t_2}} := f_{01}(t_1, t_2) + f_{11}(t_1, t_2)x_{t_2} + f_{02}(t_1, t_2)y_{t_2} + \cdots.
\]
It is easy to see that
\[
\{\pi_1^* \psi_{A_0} (\hat{f}(t_1, t_2), \hat{p}(t_1, t_2), \hat{p}(t_1))\} (f(t_1, t_2) \otimes p(t_1, t_2)^{\otimes d}) = F,
\]
\[
\{\pi_1^* \psi_{A_1} (\hat{f}(t_1, t_2), \hat{p}(t_1, t_2), \hat{p}(t_1))\} (f(t_1, t_2) \otimes p(t_1, t_2)^{\otimes d} \otimes v) = F_{x_{t_2}},
\]
\[
\{\pi_1^* \psi_{A_1} (\hat{f}(t_1, t_2), \hat{p}(t_1, t_2), \hat{p}(t_1))\} (f(t_1, t_2) \otimes p(t_1, t_2)^{\otimes d} \otimes w) = F_{y_{t_2}}.
\]
We now observe that (6.5) has a solution if and only if the following set of equations has a solution

\[ F = 0, \quad F_{x_{t_2}} = 0, \quad F_{y_{t_2}} = 0, \quad (x_{t_2}, y_{t_2}) \neq (0, 0) \quad (\text{but small}). \]  

(6.7)

Note that in Eq. (6.5) the equality holds as functionals, while in Eq. (6.7), the equality holds as numbers. We will now show that (6.7) (and hence (6.5)) has solutions whenever \((\tilde{f}, \tilde{p}, \tilde{p}) \in \Delta(A_1 \cup A_2)\). Furthermore, for all those solutions, (6.6) holds.

First let us assume \((\tilde{f}, \tilde{p}, \tilde{p}) \in \Delta A_1\). It is obvious that solutions to (6.7) exist; we can solve for \(f_{10}(t_1, t_2)\) and \(f_{01}(t_1, t_2)\) using \(F_{x_{t_2}} = 0\) and \(F_{y_{t_2}} = 0\) and then solve for \(f_{00}(t_1, t_2)\) using \(F = 0\). To show that (6.6) holds it suffices to show that if \((x_{t_2}, y_{t_2})\) is small but non-zero, then

\[(f_{00}(t_1, t_2), f_{10}(t_1, t_2), f_{01}(t_1, t_2)) \neq (0, 0, 0).\]  

(6.8)

Observe that (6.7) implies that

\[
\begin{pmatrix}
  f_{10}(t_1, t_2) \\
  f_{01}(t_1, t_2)
\end{pmatrix} = - \begin{pmatrix}
  f_{20}(t_1, t_2) & f_{11}(t_1, t_2) \\
  f_{11}(t_1, t_2) & f_{02}(t_1, t_2)
\end{pmatrix} \begin{pmatrix} x_{t_2} \\ y_{t_2} \end{pmatrix} + \begin{pmatrix} E_1(x_{t_2}, y_{t_2}) \\ E_2(x_{t_2}, y_{t_2}) \end{pmatrix}
\]  

(6.9)

where \(E_i(x_{t_2}, y_{t_2})\) are second order in \((x_{t_2}, y_{t_2})\). Since \((\tilde{f}, \tilde{p}, \tilde{p}) \in \Delta A_1\), the matrix

\[
M := \begin{pmatrix}
  f_{20}(t_1, t_2) & f_{11}(t_1, t_2) \\
  f_{11}(t_1, t_2) & f_{02}(t_1, t_2)
\end{pmatrix}
\]

is invertible if \(\tilde{f}(t_1, t_2)\) is sufficiently close to \(\tilde{f}\). Eq. (6.9) now implies that if \((x_{t_2}, y_{t_2})\) is small but non-zero, then \(f_{10}(t_1, t_2)\) and \(f_{01}(t_1, t_2)\) cannot both be zero. Therefore, (6.8) holds and hence (6.6) holds.

Next, let \((\tilde{f}, \tilde{p}, \tilde{p}) \in \Delta A_2\). Since \(\Delta A_2 \subset \Delta \bar{A}_1\), we conclude that solutions to (6.5) exist; we only need to show that (6.6) holds. Observe that \(f_{20}\) and \(f_{02}\) cannot both be zero; assume \(f_{02} \neq 0\) (and hence \(f_{02}(t_1, t_2) \neq 0\)). Write \(F\) as

\[
F = f_{00}(t_1, t_2) + f_{10}(t_1, t_2)x_{t_2} + f_{01}(t_1, t_2)y_{t_2} + A_0(x_{t_2}) + A_1(x_{t_2})y_{t_2} + A_2(x_{t_2})y_{t_2}^2 + \cdots
\]

where \(A_2(0) \neq 0\). We claim that there exists a unique holomorphic function \(B(x_{t_2})\), vanishing at the origin, such that after we make a change of coordinates \(y_{t_2} = \hat{y}_{t_2} + B(x_{t_2})\), the function \(F\) is given by

\[
F = f_{00}(t_1, t_2) + f_{10}(t_1, t_2)x_{t_2} + f_{01}(t_1, t_2)\hat{y}_{t_2} + f_{01}(t_1, t_2)B(x_{t_2}) + \hat{A}_0(x_{t_2})
\]

\[
+ \hat{A}_2(x_{t_2})\hat{y}_{t_2}^2 + \hat{A}_3(x_{t_2})\hat{y}_{t_2}^3 + \cdots
\]

for some \(\hat{A}_k(x_{t_2})\) (i.e., \(\hat{A}_1(x_{t_2}) \equiv 0\)). This is possible if \(B(x_{t_2})\) satisfies the identity
\[ A_1(x_{t_2}) + 2A_2(x_{t_2})B(x_{t_2}) + 3A_3(x_{t_2})B(x_{t_2})^2 + \cdots \equiv 0. \] (6.10)

Since \( A_2(0) \neq 0 \), \( B(x_{t_2}) \) exists by the Implicit Function Theorem and we can compute \( B(x_{t_2}) \) explicitly as a power series using (6.10) and then compute \( \hat{A}_0(x_{t_2}) \). Hence,

\[
F = f_{00}(t_1, t_2) + f_{10}(t_1, t_2)x_{t_2} + f_{01}(t_1, t_2)\hat{y}_{t_2} + \varphi(x_{t_2}, \hat{y}_{t_2})\hat{y}_{t_2}^2 + f_{01}(t_1, t_2)B(x_{t_2}) \\
\quad + \frac{B^f_{2}(t_1, t_2)}{2!}x_{t_2}^2 + \frac{B^f_{3}(t_1, t_2)}{3!}x_{t_2}^3 + O(x_{t_2}^4),
\]

where \( \varphi(0, 0) \neq 0 \) and

\[
B^f_{2}(t_1, t_2) := f_{20}(t_1, t_2) - \frac{f_{11}(t_1, t_2)^2}{f_{02}(t_1, t_2)}, \quad \text{and} \\
B^f_{3}(t_1, t_2) := f_{30}(t_1, t_2) - \frac{3f_{11}(t_1, t_2)f_{21}(t_1, t_2)}{f_{02}(t_1, t_2)} + \frac{3f_{11}(t_1, t_2)^2f_{12}(t_1, t_2)}{f_{02}(t_1, t_2)^2} \\
\quad - \frac{f_{11}(t_1, t_2)^3}{f_{02}(t_1, t_2)^3} \neq 0.
\]

The last inequality holds because \( (\hat{f}, \hat{p}, \hat{p}) \in \Delta A_2 \). In these new coordinates \( \hat{y}_{t_2} \) and \( x_{t_2} \), Eq. (6.7) is equivalent to

\[
F = f_{00}(t_1, t_2) + f_{10}(t_1, t_2)x_{t_2} + f_{01}(t_1, t_2)\hat{y}_{t_2} + \varphi(x_{t_2}, \hat{y}_{t_2})\hat{y}_{t_2}^2 + f_{01}(t_1, t_2)B(x_{t_2}) \\
\quad + \frac{B^f_{2}(t_1, t_2)}{2!}x_{t_2}^2 + \frac{B^f_{3}(t_1, t_2)}{3!}x_{t_2}^3 + O(x_{t_2}^4) = 0, \quad (6.11)
\]

\[
f_{10}(t_1, t_2) + f_{01}(t_1, t_2)B'(x_{t_2}) + \varphi_{x_{t_2}}(x_{t_2}, \hat{y}_{t_2})\hat{y}_{t_2}^2 + B^f_{2}(t_1, t_2)x_{t_2} \\
\quad + \frac{B^f_{3}(t_1, t_2)}{2!}x_{t_2}^2 + O(x_{t_2}^3) = 0, \quad (6.12)
\]

\[
f_{01}(t_1, t_2) + 2\hat{y}_{t_2}\varphi(x_{t_2}, \hat{y}_{t_2}) + \hat{y}_{t_2}^2\varphi_{x_{t_2}}(x_{t_2}, \hat{y}_{t_2}) = 0, \\
\quad (x_{t_2}, \hat{y}_{t_2}) \neq (0, 0) \quad \text{but small}. \quad (6.13)
\]

Let us clarify a point of confusion: we are claiming that (6.7) has a solution if and only if the equation

\[
F = 0, \quad F_{x_{t_2}} = 0, \quad F_{\hat{y}_{t_2}} = 0, \quad (x_{t_2}, \hat{y}_{t_2}) \neq (0, 0) \quad \text{(but small)} \quad (6.14)
\]

has a solution. We are not claiming that the partial derivatives in the old coordinates are individually equal to the partial derivatives in the new coordinates. Since \( \varphi(0, 0) \neq 0 \) we can use (6.13) to solve for \( \hat{y}_{t_2} \) in terms of \( x \) and \( f_{01}(t_1, t_2) \) to get
\[ \hat{y}_{t_2} = f_{01}(t_1, t_2) E(x_{t_2}, f_{01}(t_1, t_2)), \]

(6.15)

where \( E(x_{t_2}, f_{01}(t_1, t_2)) \) is a holomorphic function of \((x_{t_2}, f_{01}(t_1, t_2))\). Using (6.15), (6.12) and (6.11) we get (by eliminating \( B_{3}^{f(t_1, t_2)} \) and \( \hat{y}_{t_2} \))

\[
\begin{align*}
F &= - \frac{B_{3}^{f(t_1, t_2)}}{12} x_{t_2}^3 + O(x_{t_2}^4) + f_{00}(t_1, t_2) + f_{10}(t_1, t_2) E_{1}(x_{t_2}, f_{10}(t_1, t_2), f_{01}(t_1, t_2)) \\
& \quad + f_{01}(t_1, t_2) E_{2}(x_{t_2}, f_{10}(t_1, t_2), f_{01}(t_1, t_2)) = 0
\end{align*}
\]

(6.16)

where \( E_{i}(x_{t_2}, f_{10}(t_1, t_2), f_{01}(t_1, t_2)) \) is a holomorphic function of \((x_{t_2}, f_{10}(t_1, t_2), f_{01}(t_1, t_2))\). Since solutions to Eq. (6.7) satisfy (6.16), we conclude that \( f_{00}(t_1, t_2) \), \( f_{10}(t_1, t_2) \) and \( f_{01}(t_1, t_2) \) cannot all be zero. If they were all zero then \( F \) could not be zero for small but non-zero \( x_{t_2} \) (by (6.16)), since \( B_{3}^{f(t_1, t_2)} \neq 0 \) (this is where we are using \((\tilde{f}, \tilde{p}, \tilde{p}) \in \Delta A_{2}\)). This implies (6.6) holds as functionals. This completes the proof of Lemma 6.3 (1). \( \square \)

Before proceeding to the proof of Lemma 6.3 (2), let us prove a corollary which will be needed in the proof of Eq. (3.2) in Section 7. The proof of this corollary follows from the setup of the preceding proof, hence we prove it here.

**Corollary 6.6.** Let \( \mathcal{W} \rightarrow \mathcal{D} \times \mathbb{P}^2 \times \mathbb{P}^2 \) be a vector bundle such that the rank of \( \mathcal{W} \) is same as the dimension of \( \Delta A_{2} \) and \( \mathcal{Q} : \mathcal{D} \times \mathbb{P}^2 \times \mathbb{P}^2 \rightarrow \mathcal{W} \) a generic smooth section. Then the contribution of the section

\[
\pi_{3}^{*} \psi_{A_{0}} \oplus \pi_{2}^{*} \psi_{A_{1}} \oplus \mathcal{Q} : \mathcal{A}_{1} \times \mathbb{P}^2 \longrightarrow \pi_{3}^{*} \mathcal{L}_{A_{0}} \oplus \pi_{2}^{*} \mathcal{V}_{A_{1}} \oplus \mathcal{W}
\]

to the Euler class from the points of \( \Delta A_{1} \) is given by

\[
\mathcal{C}_{\Delta A_{1}}(\pi_{3}^{*} \psi_{A_{0}} \oplus \pi_{2}^{*} \psi_{A_{1}} \oplus \mathcal{Q}) = \langle e(\pi_{2}^{*} \mathcal{L}_{A_{0}} \oplus \mathcal{W}), [\Delta \mathcal{A}_{1}] \rangle.
\]

(6.17)

Secondly, if \((\tilde{f}, \tilde{p}, \tilde{p}) \in \Delta A_{2} \cap \mathcal{Q}^{-1}(0)\), then this section vanishes on \((\tilde{f}, \tilde{p}, \tilde{p})\) with a multiplicity of 3.

**Remark 6.7.** When we use the phrase “number of zeros” (resp. “number of solutions”) our intended meaning is number of zeros counted with a sign (resp. the number of solutions counted with a sign).

**Proof of Corollary 6.6.** The contribution of \( \pi_{3}^{*} \psi_{A_{0}} \oplus \pi_{2}^{*} \psi_{A_{1}} \oplus \mathcal{Q} \) to the Euler class from the points of \( \Delta A_{1} \) is the number of solutions of

---

\( ^{9} \) This dimension is also one less than the dimension of \( \Delta A_{1} \), equal to \( \delta_{A} - 2 \).
\[ \pi_2^* \psi_{A_0}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), \tilde{p}(t_1)) + \nu_0(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), \tilde{p}(t_1)) = 0, \]
\[ \pi_2^* \psi_{A_1}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), \tilde{p}(t_1)) + \nu_1(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), \tilde{p}(t_1)) = 0, \]
\[ Q(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), \tilde{p}(t_1)) = 0, \quad (\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), \tilde{p}(t_1)) \in \mathcal{U}_K \subset \overline{A}_1 \times \mathbb{P}^2, \quad (6.18) \]

where $\mathcal{K}$ is a sufficiently large compact subset of $\Delta A_1$, $\mathcal{U}_K$ is a sufficiently small neighborhood of $\mathcal{K}$ inside $\overline{A}_1 \times \mathbb{P}^2$, and $\nu_0$ and $\nu_1$ are generic smooth perturbations. We do not need to perturb $Q$ since it is already generic. We will convert the functional equation (6.18) into an equation that involves equality of numbers. Let $\theta_1, \theta_2, \ldots, \theta_{\delta_d-2}$ form a basis for $\mathcal{W}_r^*$ at the point $(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), \tilde{p}(t_1))$. Define

\[ \xi_0(\tilde{f}(t_1, t_2), \tilde{p}(t_1), (x_{t_2}, y_{t_2})) \]
\[ := \{ \nu_0(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), \tilde{p}(t_1)) \} (f(t_1, t_2) \otimes p(t_1)^{\otimes d}) \in \mathbb{C}, \]
\[ \xi_{1x}(\tilde{f}(t_1, t_2), \tilde{p}(t_1), (x_{t_2}, y_{t_2})) \]
\[ := \{ \nu_1(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), \tilde{p}(t_1)) \} (f(t_1, t_2) \otimes p(t_1)^{\otimes d} \otimes v) \in \mathbb{C}, \]
\[ \xi_{1y}(\tilde{f}(t_1, t_2), \tilde{p}(t_1), (x_{t_2}, y_{t_2})) \]
\[ := \{ \nu_1(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), \tilde{p}(t_1)) \} (f(t_1, t_2) \otimes p(t_1)^{\otimes d} \otimes w) \in \mathbb{C}, \]
\[ \mathcal{R}(\tilde{f}(t_1, t_2), \tilde{p}(t_1), (x_{t_2}, y_{t_2})) \]
\[ := \{ Q(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), \tilde{p}(t_1)) \} (\theta_1 \oplus \theta_2 \oplus \cdots \oplus \theta_{\delta_d-2}) \in \mathbb{C}^{\delta_d-2}. \quad (6.19) \]

Consider now the following set of equations

\[ \xi_0(\tilde{f}(t_1, t_2), \tilde{p}(t_1), (x_{t_2}, y_{t_2})) \]
\[ + x_{t_2} \xi_{1x}(\tilde{f}(t_1, t_2), \tilde{p}(t_1), (x_{t_2}, y_{t_2})) \]
\[ + y_{t_2} \xi_{1y}(\tilde{f}(t_1, t_2), \tilde{p}(t_1), (x_{t_2}, y_{t_2})) \]
\[ + \frac{f_{20}(t_1, t_2)}{2} x_{t_2} + f_{11}(t_1, t_2) x_{t_2} y_{t_2} + \frac{f_{20}(t_1, t_2)}{2} y_{t_2}^2 + \cdots = 0 \quad (6.20) \]
\[ \mathcal{R}(\tilde{f}(t_1, t_2), \tilde{p}(t_1), (x_{t_2}, y_{t_2})) = 0 \quad (6.21) \]
\[ \xi_{1x}(\tilde{f}(t_1, t_2), \tilde{p}(t_1), (x_{t_2}, y_{t_2})) + f_{20}(t_1, t_2) x_{t_2} + f_{11}(t_1, t_2) y_{t_2} + \cdots = 0 \quad (6.22) \]
\[ \xi_{1y}(\tilde{f}(t_1, t_2), \tilde{p}(t_1), (x_{t_2}, y_{t_2})) + f_{11}(t_1, t_2) x_{t_2} + f_{20}(t_1, t_2) y_{t_2} + \cdots = 0 \quad (6.23) \]
\[ (x_{t_2}, y_{t_2}) = \text{small}, \quad |f_{20}(t_1, t_2) f_{20}(t_1, t_2) - f_{11}(t_1, t_2)^2| > C \quad (6.24) \]

where $C$ is a small positive constant. Observe that the number of solutions of (6.18) is same as the number of solutions of (6.20)–(6.24). Let $N$ be the number of solutions $(\tilde{f}, \tilde{p})$ of

\[ \xi_0(\tilde{f}, \tilde{p}, 0, 0) = 0, \quad \mathcal{R}(\tilde{f}, \tilde{p}, 0, 0) = 0, \quad f_{00} = 0, \quad (f_{10}, f_{01}) = (0, 0). \quad (6.25) \]
Observe that this number is same as the number of solutions

\[ \nu_0(\hat{f}, \hat{p}, \hat{p}) = 0, \quad Q(\hat{f}, \hat{p}, \hat{p}) = 0, \quad (\hat{f}, \hat{p}, \hat{p}) \in \Delta \mathcal{A}_1. \]  

(6.26)

Hence

\[ N = \langle e(\pi_2^* \mathcal{L}_{A_0} \oplus \mathcal{W}), [\Delta \mathcal{A}_1] \rangle. \]  

(6.27)

Since \( Q \) is generic, all solutions of (6.26) (and hence (6.25)) belong to \( \Delta A_1 \), i.e.,

\[ f_{20}f_{02} - f_{11}^2 \neq 0 \quad \Rightarrow \quad |f_{20}f_{02} - f_{11}^2| > C. \]  

(6.28)

Let \((\hat{f}, \hat{p})\) be a solution of (6.25). Since the sections

\[ \left( \pi_2^* \psi_{A_0} + \nu_0 \right) \oplus \mathcal{Q} : \mathcal{A}_0 \times \{ \hat{p} \} \rightarrow \pi_2^* \mathcal{L}_{A_0} \oplus \mathcal{W}, \]

\[ \left( \pi_2^* \psi_{A_1} + \nu_0 \right) \oplus \mathcal{Q} : \mathcal{A}_0 \times \mathbb{P}^2 \rightarrow \pi_2^* \mathcal{L}_{A_0} \oplus \mathcal{W} \]

are transverse to the zero set (for any \( \hat{p} \in \mathbb{P}^2 \)) we conclude that given an \((x_{t_2}, y_{t_2})\) sufficiently small, there exists a unique \((\hat{f}(t_1, t_2), \hat{p}(t_1)) \in \mathcal{A}_0\) close to \((\hat{f}, \hat{p})\), such that \((\hat{f}(t_1, t_2), \hat{p}(t_1), (x_{t_2}, y_{t_2}))\) solves (6.20) and (6.21). Plugging this value in (6.22) and (6.23) and using (6.28), we conclude there is a unique \((x_{t_2}, y_{t_2})\) that solves (6.22) and (6.23) (provided the norms of \(\xi_{1x}\) and \(\xi_{1y}\) are sufficiently small). Hence, there is a one to one correspondence between the number solutions of (6.25) and the solutions of (6.20)–(6.24). Eq. (6.27) now proves (6.17).

Next, suppose \((\hat{f}, \hat{p}, \hat{p}) \in \Delta A_2 \cap \mathcal{Q}^{-1}(0)\). Since \(f_{20}\) and \(f_{02}\) are not both zero, let us assume \(f_{02} \neq 0\). The contribution of the section

\[ \pi_2^* \psi_{A_0} \oplus \pi_2^* \psi_{A_1} \oplus \mathcal{Q} : \mathcal{A}_1 \times \mathbb{P}^2 \rightarrow \pi_2^* \mathcal{L}_{A_0} \oplus \pi_2^* \mathcal{V}_{A_1} \oplus \mathcal{W} \]

to the Euler class is the number of solutions of

\[ \pi_2^* \psi_{A_0}(\hat{f}(t_1, t_2), \hat{p}(t_1, t_2), \hat{p}(t_1)) = \nu_0, \quad \pi_2^* \psi_{A_1}(\hat{f}(t_1, t_2), \hat{p}(t_1, t_2), \hat{p}(t_1)) = \nu_1 \]

\[ \mathcal{Q}(\hat{f}(t_1, t_2), \hat{p}(t_1, t_2), \hat{p}(t_1)) = 0, \quad (\hat{f}(t_1, t_2), \hat{p}(t_1, t_2), \hat{p}(t_1)) \in \mathcal{U}(\hat{f}, \hat{p}, \hat{p}) \subset \mathcal{A}_1 \times \mathbb{P}^2 \]

(6.29)

where \(\mathcal{U}(\hat{f}, \hat{p}, \hat{p})\) is a sufficiently small neighbourhood of \((\hat{f}, \hat{p}, \hat{p})\) inside \(\mathcal{A}_1 \times \mathbb{P}^2\) and \(\nu_0\) and \(\nu_1\) are generic smooth perturbations. Let \(\xi_0, \xi_{1x}, \xi_{1y}\) and \(\mathcal{R}\) be defined as in (6.19). The number of solutions of (6.29) (which is a functional equation), is equal to the number of solutions of (6.20)–(6.23) and

\[ (x_{t_2}, y_{t_2}) = \text{small}, \]

\[ f_{02}(t_1, t_2)B_2^f(t_1, t_2) := f_{02}(t_1, t_2)f_{20}(t_1, t_2) - f_{11}(t_1, t_2) = \text{small}. \]  

(6.30)
Since the section
\[ \psi_{A_2} \oplus Q : \Delta \overline{A}_1 \rightarrow \mathcal{L}_{A_2} \oplus \mathcal{W} \]
is transverse to the zero set, we conclude that there is a unique \((\hat{f}(t_1, t_2), \hat{p}(t_1)) \in \Delta \overline{A}_1 \cap Q^{-1}(0)\) close to \((\hat{f}, \hat{p})\) with a specified value of \(B_2^{f(t_1, t_2)}\). In other words, we can express all the \(f_{ij}(t_1, t_2)\) in terms of \(B_2^{f(t_1, t_2)}\). Plug in this expression for \(f_{ij}(t_1, t_2)\) in (6.20), (6.22) and (6.23). Since \((\hat{f}(t_1, t_2), \hat{p}(t_1, t_2), \hat{p}(t_1))\) is close to \(\Delta A_2\), we conclude that after a change of coordinates, the set of Eqs. (6.20), (6.22) and (6.23) is equivalent to (6.11), (6.12) and (6.13), with
\[ f_{00}(t_1, t_2) = \xi_0, \quad f_{10}(t_1, t_2) = \xi_{1x}, \quad f_{01}(t_1, t_2) = \xi_{1y}. \quad (6.31) \]

Hence, (6.16) holds which combined with (6.31) implies that the multiplicity is 3 to one in \(x_{t_2}\). Given \(x_{t_2}\), we can solve for \(y_{t_2}\) uniquely using (6.15). And given \(x_{t_2}\) and \(y_{t_2}\), we can uniquely solve for \(B_2^{f(t_1, t_2)}\) using (6.12), provided \(\xi_{1x}\) and \(\xi_{1y}\) are sufficiently small. Hence, the total multiplicity is 3. \(\square\)

**Proof of Lemma 6.3 (2).** It suffices to show that
\[ \{(\hat{f}, \hat{p}, l_{\hat{p}}) \in \overline{A}_1 \circ \hat{A}_1^\# \} = \Delta \overline{A}_3. \quad (6.32) \]

By using Lemma 5.2 with Lemma 6.2 (1) we get \(\overline{A}_1 = \hat{A}_1 \cup \overline{A}_2\). Now we use Lemma 5.2 again with Lemma 6.1 (8) to get \(\overline{A}_2 = \hat{A}_2 \cup \overline{A}_3 \cup \overline{D}_4\). By Lemmas 6.1 (9) and 5.2, we conclude that \(\overline{D}_4\) is a subset of \(\overline{A}_3\). Hence \(\overline{A}_2 = \hat{A}_2 \cup \overline{A}_3\). This implies that
\[ \Delta \overline{A}_1 = \Delta \hat{A}_1 \cup \Delta \hat{A}_2 \cup \Delta \overline{A}_3. \]

First we observe that the lhs of (6.32) is a subset of its rhs. To see this, observe that by (6.4)
\[ \{(\hat{f}, \hat{p}, l_{\hat{p}}) \in \overline{A}_1 \circ \hat{A}_1\} \cap \Delta (A_1 \cup A_2) = \emptyset \]
\[ \Rightarrow \{(\hat{f}, \hat{p}, l_{\hat{p}}) \in \overline{A}_1 \circ \hat{A}_1\} \cap \Delta (\hat{A}_1 \cup \hat{A}_2) = \emptyset \]
\[ \Rightarrow \{(\hat{f}, \hat{p}, l_{\hat{p}}) \in \overline{A}_1 \circ \hat{A}_1^\#\} \cap \Delta (\hat{A}_1 \cup \hat{A}_2) = \emptyset \quad \text{since } \overline{A}_1^\# = \overline{A}_1 \quad \text{(Lemma 6.1 (2))}. \]

Next we will show that the rhs of (6.32) is a subset of its lhs. We will simultaneously prove the following two statements.

---

\(^{10}\) This is true provided \(B_2^{f(t_1, t_2)}\) is sufficiently small.
\[
\{(f, \tilde{p}, l_{\tilde{p}}) \in \overline{\mathcal{A}_1 \circ \hat{\mathcal{A}}_1^\#} \} \supset (\hat{\mathcal{A}}_3 \sqcup \hat{\mathcal{D}}_4^\#),
\]
\[
\{(f, \tilde{p}, l_{\tilde{p}}) \in \overline{\mathcal{A}_1 \circ \mathcal{P} \mathcal{A}_2} \} \cap (\hat{\mathcal{A}}_3 \sqcup \hat{\mathcal{D}}_4^\#) = \emptyset.
\]

Since \( \overline{\mathcal{A}_1 \circ \hat{\mathcal{A}}_1^\#} \) is a closed set, (6.33) implies that the rhs of (6.32) is a subset of its lhs.\(^{11}\) This completes the proof. \(\square\)

**Claim 6.8.** Let \((f, \tilde{p}, l_{\tilde{p}}) \in (\hat{\mathcal{A}}_3 \sqcup \hat{\mathcal{D}}_4^\#)\). Then there exist solutions
\[
(f(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) \in (D \times \mathbb{P}^2) \circ \hat{\mathcal{A}}_1^\#
\]
close to \((f, \tilde{p}, l_{\tilde{p}})\) to the set of equations
\[
\pi_1^* \psi_{\mathcal{A}_0} (f(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) = 0,
\]
\[
\pi_1^* \psi_{\mathcal{A}_1} (f(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) = 0, \quad \tilde{p}(t_1, t_2) \neq \tilde{p}(t_1).
\]
Moreover, whenever such a solution is sufficiently close to \((f, \tilde{p}, l_{\tilde{p}})\), it lies in \(\mathcal{A}_1 \circ \hat{\mathcal{A}}_1^\#\), i.e.,
\[
\pi_2^* \Psi_{\mathcal{P} \mathcal{A}_2} (f(t_1, t_2), \tilde{p}(t_1, t_2), \tilde{p}(t_1)) \neq 0.
\]
In particular, \((f(t_1, t_2), \tilde{p}(t_1, t_2), \tilde{p}(t_1))\) does not lie in \(\mathcal{A}_1 \circ \mathcal{P} \mathcal{A}_2\).

It is easy to see that Claim 6.8 proves (6.33) and (6.34) simultaneously.

**Proof.** Choose homogeneous coordinates \([X : Y : Z]\) so that \(\tilde{p} = [0 : 0 : 1]\) and let \(\mathcal{U}_{\tilde{p}}, \pi_x, \pi_y, v, w, x_{t_1}, y_{t_1}, x_{t_2}, y_{t_2}, f_{ij}(t_1, t_2)\) be exactly the same as defined in the proof of Claim 6.5. Take
\[
(f(t_1, t_2), l_{\tilde{p}(t_1)}) \in \hat{\mathcal{A}}_1^\#
\]
to be a point that is close to \((f, \tilde{p})\) and \(l_{\tilde{p}(t_1, t_2)}\) a point in \(\mathbb{P} \mathbb{P}^2\) that is close to \(l_{\tilde{p}(t_1)}\). Without loss of generality, we can assume that
\[
v + \eta w \in l_{\tilde{p}}, \quad v + \eta_{t_1} w \in l_{\tilde{p}(t_1)} \quad \text{and} \quad v + (\eta_{t_1} + \eta_{t_2}) w \in l_{\tilde{p}(t_1, t_2)}
\]
for some complex numbers \(\eta, \eta_{t_1}\) and \(\eta_{t_1} + \eta_{t_2}\) close to each other. Let the numbers \(F, F_{x_{t_2}}\) and \(F_{y_{t_2}}\) be the same as in the proof of Claim 6.5. Since \((f(t_1, t_2), l_{\tilde{p}(t_1)}) \in \hat{\mathcal{A}}_1^\#\), we conclude that
\[
f_{00}(t_1, t_2) = f_{10}(t_1, t_2) = f_{01}(t_1, t_2) = 0.
\]
\(^{11}\) As before, we do not need the full strength of (6.33). We simply need that \(\{(f, \tilde{p}, l_{\tilde{p}}) \in \overline{\mathcal{A}_1 \circ \hat{\mathcal{A}}_1^\#} \} \supset \Delta \mathcal{A}_3.\)
The functional equation (6.35) has a solution if and only if the following has a numerical solution:

\[ F = 0, \quad F_{x_{t_2}} = 0, \quad F_{y_{t_2}} = 0, \quad (x_{t_2}, y_{t_2}) \neq (0, 0) \quad (\text{but small}). \quad (6.37) \]

First let us assume \((\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \Delta \hat{A}_3\). It is easy to see that \(f_{00}\) and \(f_{02}\) cannot both be zero; let us assume \(f_{02}\) is non-zero. Following the same argument as in the proof of Claim 6.5, we make a change of coordinates and write \(F\) as

\[
F = \frac{\hat{y}_{t_2}^2}{2!} x_{t_2}^2 + \frac{B_2^{f(t_1, t_2)}}{3!} x_{t_2}^3 + \frac{B_4^{f(t_1, t_2)}}{4!} x_{t_2}^4 + O(x_{t_2}^5),
\]

where \(\hat{y}_{t_2} := \sqrt{\varphi(x_{t_2}, \hat{y}_{t_2})}\hat{y}_{t_2}, \quad B_2^{f(t_1, t_2)} = f_{20}(t_1, t_2) - \frac{f_{11}(t_1, t_2)^2}{f_{02}(t_1, t_2)}. \quad (6.37)\)

It is easy to see that the only solutions to (6.37) (in terms of the new coordinates) are

\[
B_2^{f(t_1, t_2)} = \frac{B_4^{f(t_1, t_2)}}{12} x_{t_2}^2 + O(x_{t_2}^3), \quad B_3^{f(t_1, t_2)} = -\frac{B_4^{f(t_1, t_2)}}{2} x_{t_2} + O(x_{t_2}^2),
\]

\[
\hat{y}_{t_2} = 0, \quad x_{t_2} \neq 0 \quad (\text{but small}). \quad (6.38)\]

It remains to show that these solutions satisfy (6.36). First consider the case when \((\tilde{f}, \tilde{p}, l_{\tilde{p}}) \notin \mathcal{PA}_3\), i.e., \(\pi_2^\star \Psi_{\mathcal{PA}_2}(\tilde{f}, \tilde{p}, l_{\tilde{p}}) \neq 0\). Then (6.36) is obviously true, since the section \(\pi_2^\star \Psi_{\mathcal{PA}_2}\) is continuous. Next, consider the case when \((\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \mathcal{PA}_3\). Define the numbers

\[
J_1 := f_{20}(t_1, t_2) + \eta_1 f_{11}(t_1, t_2), \quad J_2 := f_{11}(t_1, t_2) + \eta_1 f_{02}(t_1, t_2). \quad (6.39)\]

Observe that

\[
\{ \pi_2^\star \Psi_{\mathcal{PA}_2}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) \} (f(t_1, t_2) \otimes p(t_1)^{\otimes d} \otimes (v + \eta_1 w) \otimes v) = J_1,
\]

\[
\{ \pi_2^\star \Psi_{\mathcal{PA}_2}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) \} (f(t_1, t_2) \otimes p(t_1)^{\otimes d} \otimes (v + \eta_1 w) \otimes w) = J_2.
\]

\[
(6.40)\]

If (6.36) were false, then \(J_1\) and \(J_2\) would vanish (by (6.40)). Eqs. (6.39) and (6.38) imply

\[
\frac{B_4^{f(t_1, t_2)}}{12} x_{t_2}^2 + O(x_{t_2}^3) = J_1 - \frac{f_{11}(t_1, t_2)}{f_{02}(t_1, t_2)} J_2. \quad (6.41)\]

Since \((\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \hat{A}_3\), we conclude \(B_4^{f(t_1, t_2)} \neq 0\). Hence, (6.41) implies that \(J_1\) and \(J_2\) cannot both be zero; consequently (6.36) holds.
Next, let us assume that \((\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \Delta \hat{D}_4^{\#b}\). Define the following number

\[
G := x_{t_2} F_{x_{t_2}} + y_{t_2} F_{y_{t_2}} - 2F
\]

\[
= \frac{f_{30}(t_1, t_2)}{6} x_{t_2}^3 + \frac{f_{21}(t_1, t_2)}{2} x_{t_2}^2 y_{t_2} + \frac{f_{12}(t_1, t_2)}{2} x_{t_2} y_{t_2}^2 + \frac{f_{03}(t_1, t_2)}{6} y_{t_2}^3 + \cdots.
\]

Note that the cubic term of \(G\) is same as the cubic term of \(F\). Since \((\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \Delta \hat{D}_4\) we conclude that there exists a change of coordinates

\[
x_{t_2} = \hat{x}_{t_2} + E_1(\hat{x}_{t_2}, \hat{y}_{t_2}), \quad y_{t_2} = \hat{y}_{t_2} + E_2(\hat{x}_{t_2}, \hat{y}_{t_2}),
\]

(\(E_i(\hat{x}_{t_2}, \hat{y}_{t_2})\) are second order in \(\hat{x}_{t_2}\) and \(\hat{y}_{t_2}\)) so that \(G\) is given by

\[
G = \frac{f_{30}(t_1, t_2)}{6} \hat{x}_{t_2}^3 + \frac{f_{21}(t_1, t_2)}{2} \hat{x}_{t_2}^2 \hat{y}_{t_2} + \frac{f_{12}(t_1, t_2)}{2} \hat{x}_{t_2} \hat{y}_{t_2}^2 + \frac{f_{03}(t_1, t_2)}{6} \hat{y}_{t_2}^3. \tag{6.42}
\]

This is same as the argument in [2], where we give a necessary and sufficient criteria for a curve to have a \(D_4\)-node. Since \((\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \Delta \hat{D}_4\), there are three possibilities to consider;

\[
\begin{align*}
&f_{30}(t_1, t_2) \neq 0 \text{ or } f_{03}(t_1, t_2) \neq 0 \text{ or } \\
&f_{30}(t_1, t_2) = f_{03}(t_1, t_2) = 0, \text{ but } f_{21}(t_1, t_2) \neq 0 \text{ and } f_{12}(t_1, t_2) \neq 0. \tag{6.43}
\end{align*}
\]

Let us assume \(f_{30}(t_1, t_2) \neq 0\). Since \((\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \Delta \hat{D}_4^{\#b}\), Eq. (6.42) now can be written as

\[
G = \frac{f_{30}(t_1, t_2)}{6} (\hat{x}_{t_2} - A_1 \hat{y}_{t_2})(\hat{x}_{t_2} - A_2 \hat{y}_{t_2})(\hat{x}_{t_2} - A_3 \hat{y}_{t_2})
\]

where \(A_i\) are complex numbers such that

\[
A_1 \neq A_2 \neq A_3 \neq A_1 \quad \text{and} \quad \eta \neq A_1^{-1}, A_2^{-1} \text{ or } A_3^{-1}. \tag{6.44}
\]

To see why the last inequality is true, note that if \(\eta\) is either \(\frac{1}{A_1}\), \(\frac{1}{A_2}\) or \(\frac{1}{A_3}\) then

\[
\nabla^3 f|_{\tilde{p}}(v + \eta w, v + \eta w, v + \eta w) = 0.
\]

Since \((\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \Delta \hat{D}_4^{\#b}\) the last inequality of (6.44) holds. Hence, (6.35) has a solution if and only if

\[
G = \frac{f_{30}(t_1, t_2)}{6} (\hat{x}_{t_2} - A_1 \hat{y}_{t_2})(\hat{x}_{t_2} - A_2 \hat{y}_{t_2})(\hat{x}_{t_2} - A_3 \hat{y}_{t_2}) = 0,
\]

\[
F_{x_{t_2}} = \hat{x}_{t_2} F_{x_{t_2}} + \hat{y}_{t_2} F_{y_{t_2}} + \left(3 \hat{x}_{t_2} - 2 \hat{x}_{t_2} \hat{y}_{t_2} \left(\sum_{i=1}^{3} A_i\right) + (A_1 A_2 + A_1 A_3 + A_2 A_3) \hat{y}_{t_2}^2\right)
\]

\[
+ E_3(\hat{x}_{t_2}, \hat{y}_{t_2}) = 0,
\]
\[ F_{y_2} = \dot{x}_{t_2} f_{11}(t_1, t_2) + \dot{y}_{t_2} f_{02}(t_1, t_2) \]
\[ - \frac{f_{30}(t_1, t_2)}{6} \left( \dot{x}_{t_2}^2 \left( \sum_{i=1}^{3} A_i \right) - 2\dot{x}_{t_2} \dot{y}_{t_2} \left( \sum_{i \neq j} A_i A_j \right) + 3\dot{y}_{t_2}^2 A_1 A_2 A_3 \right) \]
\[ + E_4(\dot{x}_{t_2}, \dot{y}_{t_2}) = 0 \]
\[ (\dot{x}_{t_2}, \dot{y}_{t_2}) \neq (0, 0) \quad \text{(but small)} \quad (6.45) \]

has a solution, where \( E_i(\dot{x}_{t_2}, \dot{y}_{t_2}) \) are third order in \((\dot{x}_{t_2}, \dot{y}_{t_2})\).

To avoid confusion let us clarify one point; in the above equation \( F_{x_{t_2}} \) and \( F_{y_{t_2}} \) are simply expressed in terms of the new coordinates \( \dot{x}_{t_2} \) and \( \dot{y}_{t_2} \). They are still the partial derivatives of \( F \) with respect to \( x_{t_2} \) and \( y_{t_2} \); they are not \( F_{\dot{x}_{t_2}} \) and \( F_{\dot{y}_{t_2}} \); the partial derivatives of \( F \) with respect to \( \dot{x}_{t_2} \) and \( \dot{y}_{t_2} \). Now we will construct the solutions to (6.45). There are three solutions; we will just give one of the solutions, the rest are similar. They are given by:

\[ \dot{x}_{t_2} = A_1 \dot{y}_{t_2}, \quad \dot{y}_{t_2} \neq 0 \quad \text{(but small)}, \quad f_{20}(t_1, t_2) = \text{small}, \]
\[ f_{02}(t_1, t_2) = \frac{f_{30}(t_1, t_2)}{6} \left( 2A_1^2 - 2A_1^3 A_2 - 2A_1^2 A_3 + 2A_1 A_2 A_3 \right) \dot{y}_{t_2} \]
\[ + A_1^2 f_{20}(t_1, t_2) + E_5(\dot{y}_{t_2}), \]
\[ f_{11}(t_1, t_2) = \frac{f_{30}(t_1, t_2)}{6} \left( -A_1^2 + A_1 A_2 + A_1 A_3 - A_2 A_3 \right) \dot{y}_{t_2} \]
\[ - A_1 f_{20}(t_1, t_2) + E_6(\dot{y}_{t_2}), \quad (6.46) \]

where \( E_i(\dot{y}_{t_2}) \) are second order in \( \dot{y}_{t_2} \) and independent of \( f_{20}(t_1, t_2) \). It remains to show that (6.36) holds. Eq. (6.46) implies that

\[(1 - \eta_1, A_1) J_2 + (A_1 - A_1^2 \eta_1) J_1 = \beta (\dot{y}_{t_2}), \]
\[ \text{where} \quad \beta := -\frac{f_{30}(t_1, t_2)}{6} (A_1 - A_2)(A_1 - A_3)(-1 + A_1 \eta_1)^2, \quad (6.47) \]

and \( J_1 \) and \( J_2 \) are as defined in (6.39). Note that the rhs of (6.47) is independent of \( f_{20}(t_1, t_2) \). By (6.44), \( \beta \neq 0 \). Hence, by (6.47) \( J_1 \) and \( J_2 \) cannot both vanish. As a result, (6.36) holds. Similar argument holds for the other two solutions of (6.45).

A similar argument will go through if any of the other two cases of (6.43) holds. \( \square \)

**Corollary 6.9.** Let \( \mathbb{W} \to \mathcal{D} \times \mathbb{P}^2 \times \mathbb{P}^2 \mathcal{T}^2 \) be a vector bundle such that the rank of \( \mathbb{W} \) is equal to dimension of \( \Delta \mathcal{P} A_3 \) and \( \mathcal{Q} : \mathcal{D} \times \mathbb{P}^2 \times \mathbb{P}^2 \mathcal{T}^2 \to \mathbb{W} \) a generic smooth section. Suppose \((\bar{f}, \bar{p}, l_\bar{p}) \in \Delta \mathcal{P} A_3\). Then the section

\[ \pi_2^* \psi_{\mathcal{P} A_2} \oplus \mathcal{Q} : \overline{\mathcal{A}_1} \circ \overline{\mathcal{A}_1} \to \pi_2^* \psi_{\mathcal{P} A_2} \oplus \mathbb{W} \]

vanishes around \((\bar{f}, \bar{p}, l_\bar{p})\) with a multiplicity of 2.
Proof. Since $Q$ is generic, the sections
\[ \Psi_{\mathcal{PA}_2} \oplus Q : \Delta \overline{A}_1^\# \rightarrow \mathcal{V}_{\mathcal{PA}_2} \oplus \mathcal{W}, \quad \Psi_{\mathcal{PA}_3}^{-1} : \eta_{\mathcal{PA}_2}(0) \rightarrow \mathcal{L}_{\mathcal{PA}_3} \]
are transverse to the zero set. Hence, there exists a unique $(\hat{f}(t_1, t_2), l_{p(t_1)}) \in \Delta \overline{A}_1^\#$ close to $(\tilde{f}, l_{\tilde{p}})$ for a specified value of $J_1, J_2$ and $f_{30}(t_1, t_2).$ In other words we can express all the $f_{ij}(t_1, t_2)$ in terms of $J_1, J_2$ and $f_{30}(t_1, t_2).$ Since $B_4^{ij}(t_1, t_2) \neq 0,$ Eq. (6.41) implies that the number of solutions to the set of equations
\[ J_1 = \xi_1, \quad J_2 = \xi_2 \]
is 2, where $\xi_i$ is a small perturbation.

Corollary 6.10. Let $\mathcal{W} \rightarrow \mathcal{D} \times \mathbb{P}^2 \times \mathbb{P}T\mathbb{P}^2$ be a vector bundle such that the rank of $\mathcal{W}$ is equal to dimension of $\Delta \overline{D}_4^\#$ and $Q : \mathcal{D} \times \mathbb{P}^2 \times \mathbb{P}T\mathbb{P}^2 \rightarrow \mathcal{W}$ a generic smooth section. Suppose $(\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \Delta \overline{D}_4^\#.$ Then the section
\[ \pi_2^s \Psi_{\mathcal{PD}_4} \oplus Q : \overline{A}_1 \circ \overline{A}_4^\# \rightarrow \pi_2^s \mathcal{L}_{\mathcal{PD}_4} \oplus \mathcal{W} \]
vanishes around $(\tilde{f}, \tilde{p}, l_{\tilde{p}})$ with a multiplicity of 3.

Proof. Since $Q$ is generic, the sections
\[ \Psi_{\mathcal{PA}_2} \oplus Q : \Delta \overline{A}_1^\# \rightarrow \mathcal{V}_{\mathcal{PA}_2} \oplus \mathcal{W}, \quad \Psi_{\mathcal{PD}_4}^{-1} : \eta_{\mathcal{PA}_2}(0) \rightarrow \mathcal{L}_{\mathcal{PD}_4} \]
are transverse to the zero set. Hence, there exists a unique $(\hat{f}(t_1, t_2), l_{p(t_1)}) \in \Delta \overline{A}_1^\#$ close to $(\tilde{f}, l_{\tilde{p}})$ for a specified value of $J_1, J_2$ and $f_{02}(t_1, t_2).$ In other words we can express all the $f_{ij}(t_1, t_2)$ in terms of $J_1, J_2$ and $f_{02}(t_1, t_2).$ Since $\beta \neq 0,$ Eq. (6.47) implies that the number of solutions to the set of equations
\[ J_1 = \xi_1, \quad J_2 = \xi_2 \]
is 1, where $\xi_i$ is a small perturbation. Since there are a total of 3 solutions to (6.45), the total multiplicity is 3.

Proof of Lemma 6.3 (3). It suffices to prove the following two statements:
\[ \{(\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \overline{A}_1 \circ \mathcal{PA}_2 : \pi_2^s \Psi_{\mathcal{PD}_4}(\tilde{f}, \tilde{p}, l_{\tilde{p}}) \neq 0\} \]
\[ \{(\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \Delta \overline{PA}_2 : \pi_2^s \Psi_{\mathcal{PD}_4}(\tilde{f}, \tilde{p}, l_{\tilde{p}}) \neq 0\} = \Delta \overline{D}_5^\# \quad (6.48) \]
\[ \{(\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \overline{A}_1 \circ \mathcal{PA}_2 : \pi_2^s \Psi_{\mathcal{PD}_4}(\tilde{f}, \tilde{p}, l_{\tilde{p}}) = 0\} = \Delta \overline{D}_5^\# \quad (6.49) \]

---

\(^{12}\) Provided $J_1, J_2$ and $f_{30}(t_1, t_2)$ are sufficiently small.

\(^{13}\) Provided $J_1, J_2$ and $f_{02}(t_1, t_2)$ are sufficiently small.
Let us directly prove a more general version of (6.48):

**Lemma 6.11.** If \( k \geq 2 \), then

\[
\{(\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in A_1 \circ \mathcal{P}A_k : \pi^*_2 \Psi_{\mathcal{P}D_4}(\tilde{f}, \tilde{p}, l_{\tilde{p}}) \neq 0 \} = \{(\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \Delta \mathcal{P}A_{k+2} : \pi^*_2 \Psi_{\mathcal{P}D_4}(\tilde{f}, \tilde{p}, l_{\tilde{p}}) \neq 0 \}.
\]

Note that (6.48) is a special case of Lemma 6.11; take \( k = 2 \).

**Proof.** We will prove the following two facts simultaneously:

\[
\{(\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in A_1 \circ \mathcal{P}A_k \} \supset \Delta \mathcal{P}A_{k+2} \quad \forall k \geq 2, \tag{6.50}
\]

\[
\{(\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in A_1 \circ \mathcal{P}A_{k+1} \} \cap \Delta \mathcal{P}A_{k+2} = \emptyset \quad \forall k \geq 1. \tag{6.51}
\]

It follows from Lemma 6.2 (5) that (6.50) and (6.51) imply Lemma 6.11. We will now prove the following claim:

**Claim 6.12.** Let \((\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \Delta \mathcal{P}A_{k+2}\) and \( k \geq 2 \). Then there exists a solution

\[
(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) \in (D \times \mathbb{P}^2) \circ \mathcal{P}A_2
\]

near \((\tilde{f}, \tilde{p}, l_{\tilde{p}})\) to the set of equations

\[
\pi^*_1 \Psi_{A_0}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) = 0, \quad \pi^*_1 \Psi_{A_1}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) = 0,
\]

\[
\pi^*_2 \Psi_{\mathcal{P}A_k}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) = 0, \quad \ldots, \quad \pi^*_2 \Psi_{\mathcal{P}A_k}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) = 0,
\]

\[
\pi^*_2 \Psi_{\mathcal{P}D_4}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) \neq 0, \quad \tilde{p}(t_1, t_2) \neq \tilde{p}(t_1). \tag{6.52}
\]

Moreover, any solution \((\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)})\) sufficiently close to \((\tilde{f}, \tilde{p}, l_{\tilde{p}})\) lies in \( A_1 \circ \mathcal{P}A_k \), i.e.,

\[
\pi^*_2 \Psi_{\mathcal{P}A_{k+1}}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) \neq 0. \tag{6.53}
\]

In particular \((\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)})\) does not lie in \( A_1 \circ \mathcal{P}A_{k+1} \).

It is easy to see that Claim 6.12 implies (6.50) and (6.51) simultaneously for all \( k \geq 2 \). The fact that (6.51) holds for \( k = 1 \) follows from (6.34) (since \( \mathcal{P}A_3 \) is a subset of \( \hat{A}_3 \)).

**Proof.** Choose homogeneous coordinates \([X : Y : Z]\) so that \( \tilde{p} = [0 : 0 : 1] \) and let \( U_{\tilde{p}} \), \( \pi_x, \pi_y, x_{t_1}, y_{t_1}, x_{t_2}, y_{t_2} \) be exactly the same as defined in the proof of Claim 6.5. Let \( \psi_1, w : U_{\tilde{p}} \rightarrow T\mathbb{P}^2 \) be vectors dual to the one forms \( d\pi_x \) and \( d\pi_y \) respectively. Take

\[
(\tilde{f}(t_1, t_2), l_{\tilde{p}(t_1)}) \in \mathcal{P}A_2
\]
to be a point that is close to $(\hat{\bar{f}},l_{\bar{p}})$ and $l_{\bar{p}(t_1,t_2)}$ a point in $\mathbb{P}T\mathbb{P}^2$ that is close to $l_{\bar{p}(t_1)}$. Without loss of generality, we can assume that

$$v := v_1 + \eta w \in l_{\bar{p}}, \quad v_1 + \eta_1 w \in l_{\bar{p}(t_1)} \quad \text{and} \quad v + (\eta_1 + \eta_2) w \in l_{\bar{p}(t_1,t_2)}$$

for some complex numbers $\eta$, $\eta_1$ and $\eta_1 + \eta_2$ close to each other. Let

$$f_{ij}(t_1,t_2) := \nabla^{i+j} f(t_1,t_2)|_{p(t_1)}(v,\ldots,v,w,\ldots,w).$$

The numbers $F$, $F_{xt_2}$ and $F_{yt_2}$ are the same as in the proof of Claim 6.5. Since $(\hat{\bar{f}}(t_1,t_2),l_{\bar{p}(t_1)}) \in \mathcal{PA}_2$, we conclude that

$$f_{00}(t_1,t_2) = f_{10}(t_1,t_2) = f_{01}(t_1,t_2) = f_{20}(t_1,t_2) = f_{11}(t_1,t_2) = 0.$$

Moreover, since $(\hat{\bar{f}},l_{\bar{p}}) \in \mathcal{PA}_{k+2}$ we conclude that $f_{02}$ and $A^f_{k+3}$ are non-zero. Hence $f_{02}(t_1,t_2)$ and $A^f_{k+3}(t_1,t_2)$ are non-zero if $\hat{\bar{f}}(t_1,t_2)$ is sufficiently close to $\hat{\bar{f}}$. Since $f_{02}(t_1,t_2) \neq 0$, following the same argument as in the proof of Claim 6.5, we can make a change of coordinates to write $F$ as

$$F = \hat{y}_{t_2}^2 + \frac{A^f_{3}(t_1,t_2)}{3!} x_{t_2}^3 + \frac{A^f_{4}(t_1,t_2)}{4!} x_{t_2}^4 + \cdots$$

The functional equation (6.52) has a solution if and only if the following set of equations has a solution (as numbers):

$$\hat{y}_{t_2}^2 + \frac{A^f_{3}(t_1,t_2)}{3!} x_{t_2}^3 + \frac{A^f_{4}(t_1,t_2)}{4!} x_{t_2}^4 + \cdots = 0, \quad 2\hat{y}_{t_2} = 0,$$

$$\frac{A^f_{3}(t_1,t_2)}{2!} x_{t_2}^3 + \frac{A^f_{4}(t_1,t_2)}{3!} x_{t_2}^4 + \cdots = 0, \quad A^f_{3}(t_1,t_2), \ldots, A^f_{k}(t_1,t_2) = 0,$$

$$(\hat{y}_{t_2},x_{t_2}) \neq (0,0) \quad \text{(but small)}. \quad (6.54)$$

It is easy to see that the solutions to (6.54) exist given by

$$A^f_{3}(t_1,t_2), \ldots, A^f_{k}(t_1,t_2) = 0,$$

$$A^f_{k+1}(t_1,t_2) = \frac{A^f_{k+3}(t_1,t_2)}{(k+2)(k+3)} x_{t_2}^2 + O(x_{t_2}^3),$$

$$A^f_{k+2}(t_1,t_2) = -\frac{2A^f_{k+3}(t_1,t_2)}{(k+3)} x_{t_2} + O(x_{t_2}^2), \quad \hat{y}_{t_2} = 0, \quad x_{t_2} \neq 0 \quad \text{(but small)}. \quad (6.55)$$

By (6.55), it immediately follows that (6.53) holds (since $A^f_{k+3}(t_1,t_2) \neq 0$). $\Box$
Corollary 6.13. Let $\mathcal{W} \to \mathcal{D} \times \mathbb{P}^2 \times \mathbb{P} \mathbb{T}^2$ be a vector bundle such that the rank of $\mathcal{W}$ is same as the dimension of $\Delta \mathcal{P} A_{k+2}$ and $\mathcal{Q} : \mathcal{D} \times \mathbb{P}^2 \times \mathbb{P} \mathbb{T}^2 \to \mathcal{W}$ a generic smooth section. Suppose $(\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \Delta \mathcal{P} A_{k+2} \cap \mathcal{Q}^{-1}(0)$. Then the section

$$\pi_2^* \psi_{\mathcal{P} A_{k+1}} \oplus \mathcal{Q} : \Delta \overline{\mathcal{P} A_k} \longrightarrow \pi_2^* \mathbb{L} \mathcal{P} A_{k+1} \oplus \mathcal{W}$$

vanishes around $(\tilde{f}, \tilde{p}, l_{\tilde{p}})$ with a multiplicity of 2.

Proof. This follows from the fact that the sections

$$\pi_2^* \psi_{\mathcal{P} A_i} : \Delta \overline{\mathcal{P} A}_{i-1} - \pi_2^* \psi_{\mathcal{P} A_{D_4}}^{-1}(0) \longrightarrow \pi_2^* \mathbb{L} \mathcal{P} A_i$$

are transverse to the zero set for all $3 \leq i \leq k+2$, the fact that $\mathcal{Q}$ is generic and (6.55). The proof is now similar to that of Corollaries 6.6, 6.9 and 6.10. □

Next let us prove (6.49). First we will prove the following two facts:

$$\overline{A_1} \cap \mathcal{P} A_2 \cap \Delta \mathcal{P} D_4 = \emptyset, \quad \overline{A_1} \cap \mathcal{P} A_3 \cap \Delta \mathcal{P} D_5 = \emptyset. \quad (6.56)$$

Although (6.57) is not needed to prove (6.49), we will prove these two statements together since their proofs are very similar.

Claim 6.14. Let $(\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \Delta \mathcal{P} D_4$. Then there exist no solutions

$$(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) \in \overline{A_1} \cap \mathcal{P} A_2$$

near $(\tilde{f}, \tilde{p}, l_{\tilde{p}})$ to the set of equations

$$\pi_1^* \psi_{A_0}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) = 0,$$

$$\pi_1^* \psi_{A_1}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) = 0. \quad (6.58)$$

Secondly, let $(\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \Delta \mathcal{P} D_5$. Then there exist no solutions

$$(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) \in \overline{A_1} \cap \mathcal{P} A_3$$

near $(\tilde{f}, \tilde{p}, l_{\tilde{p}})$ to the set of equations

$$\pi_1^* \psi_{A_0}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) = 0,$$

$$\pi_1^* \psi_{A_1}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) = 0. \quad (6.59)$$
It is easy to see that Claim 6.14 proves (6.56) and (6.57).

**Proof.** For the first part, choose homogeneous coordinates \([X : Y : Z]\) so that \(\tilde{p} = [0 : 0 : 1]\) and let \(U_t, \pi_x, \pi_y, v_1, w, v, \eta, \eta_t, \eta_t, x_t, y_t, x_{t2}, y_{t2}, f_{33}(t_1, t_2), F, F_{x_{t2}}\) and \(F_{y_{t2}}\) be exactly the same as defined in the proof of Claim 6.12. Since \((\tilde{f}(t_1, t_2), l_{\tilde{p}}(t_1)) \in \tilde{\mathcal{P}}A_2\), we conclude that

\[f_{00}(t_1, t_2) = f_{10}(t_1, t_2) = f_{01}(t_1, t_2) = f_{20}(t_1, t_2) = f_{11}(t_1, t_2) = 0.\]

The functional equation (6.58) has a solution if and only if the following set of equations has a solution (as numbers):

\[F = 0, \quad F_{x_{t2}} = 0, \quad F_{y_{t2}} = 0, \quad (x_{t2}, y_{t2}) \neq (0, 0) \quad \text{(but small).}\] (6.60)

For the convenience of the reader, let us rewrite the expression for \(F\):

\[F := \frac{f_{02}(t_1, t_2)}{2} y_t^2 + \frac{f_{30}(t_1, t_2)}{6} x_t^3 + \frac{f_{21}(t_1, t_2)}{2} x_t y_t + \frac{f_{12}(t_1, t_2)}{2} x_t y_t^2 + \frac{f_{03}(t_1, t_2)}{6} y_t^3 + \cdots.\]

Since \((\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \tilde{D}_4\) we conclude that

\[f_{02} = 0, \quad f_{30} = 0, \quad f_{21} \neq 0, \quad 3f_{12}^2 - 4f_{21}f_{03} \neq 0.\] (6.61)

To see why the two non-vanishing conditions hold, first notice that since \((\tilde{f}, l_{\tilde{p}}) \in \tilde{D}_4\) we conclude that the cubic term in the Taylor expansion of \(f\) has no repeated root. In other words

\[\beta := \frac{f_{30}^2 f_{03} - 6 f_{03} f_{12} f_{21} f_{30} + 4 f_{12}^3 f_{30} + 4 f_{03} f_{21}^3 - 3 f_{12}^2 f_{21}^2}{f_{03}} \neq 0.\]

Since \((\tilde{f}, l_{\tilde{p}}) \in \tilde{D}_4\) we conclude \(f_{30} = 0\). Hence we get (6.61). Now we will show that (6.60) has no solutions. First of all we claim that \(y_{t2} \neq 0\); we will justify that at the end. Assuming that, define \(L := \frac{x_{t2}}{y_{t2}}\). Substituting \(x_{t2} = L y_{t2}\) in \(F_{x_{t2}} = 0\) and using \(y_{t2} \neq 0\) and \(f_{21}(t_1, t_2) \neq 0\) we can solve for \(L\) using the Implicit Function Theorem. That gives us

\[L = -\frac{f_{12}(t_1, t_2)}{2 f_{21}(t_1, t_2)} + y_{t2} E_1(y_{t1}, f_{30}(t_1, t_2)) + f_{30}(t_1, t_2) E_2(y_{t1}, f_{30}(t_1, t_2)).\]

(6.62)

where \(E_i(0, 0) = 0\). Using the value of \(L\) from (6.62), and substituting \(x_{t2} = L y_{t2}\) in \(F - \frac{y_{t2} f_{y_{t2}}}{2} = 0\), we conclude that as \((y_{t2}, f_{30}(t_1, t_2))\) go to zero

\[-\frac{f_{03}}{12} + \frac{f_{12}^2}{16 f_{21}} = 0.\] (6.63)
It is easy to see that (6.63) contradicts (6.61). It remains to show that \(y_{t_2} \neq 0\). To see why that is so, consider the equation \(F_{y_{t_2}} = 0\). It is easy to see that if \(y_{t_2} = 0\) then \(f_{21}(t_1, t_2)\) will go to zero, contradicting (6.61). Hence (6.60) has no solutions.

For the second part of the claim, we use the same set up except for one difference: we require \((\tilde{f}, l_{\tilde{p}}) \in \overline{PA}_3\). Hence

\[
 f_{00}(t_1, t_2) = f_{10}(t_1, t_2) = f_{01}(t_1, t_2) = f_{20}(t_1, t_2) = f_{11}(t_1, t_2) = f_{30}(t_1, t_2) = 0.
\]

The functional equation (6.59) has a solution if and only if the following set of equations has a solution (as numbers):

\[
 F = 0, \quad F_{x_{t_2}} = 0, \quad F_{y_{t_2}} = 0, \quad (x_{t_2}, y_{t_2}) \neq (0, 0) \quad \text{(but small)}. \quad (6.64)
\]

For the convenience of the reader, let us rewrite the expression for \(F\):

\[
 F := \frac{f_{02}(t_1, t_2)}{2} y_{t_2}^2 + \frac{f_{21}(t_1, t_2)}{2} x_{t_2} y_{t_2} + \frac{f_{12}(t_1, t_2)}{2} x_{t_2}^2 y_{t_2} + \frac{f_{03}(t_1, t_2)}{6} y_{t_2}^3 + \cdots.
\]

Since \((\tilde{f}, l_{\tilde{p}}) \in PD_5\) we conclude that

\[
 f_{02} = 0, \quad f_{30} = 0, \quad f_{21} = 0, \quad f_{40} \neq 0, \quad f_{12} \neq 0. \quad (6.65)
\]

We will now show that there are no solutions to (6.64). First we claim that \(y_{t_2} \neq 0\); we will justify that at the end. Assuming that, define \(L := \frac{x_{t_2}}{y_{t_2}}\). Substituting \(x_{t_2} = Ly_{t_2}\) in \(F_{x_{t_2}} = 0\) and using \(y_{t_2} \neq 0\), we conclude that as \(y_{t_2}\) and \(f_{21}(t_1, t_2)\) go to zero, \(f_{12}(t_1, t_2)\) goes to zero, contradicting (6.65). It remains to show that \(y_{t_2} \neq 0\). Consider the equation \(F_{x_{t_2}} = 0\). If \(y_{t_2} = 0\) then \(f_{40}(t_1, t_2)\) would go to zero as \(x_{t_2}\) goes to zero, contradicting (6.65). Hence (6.64) has no solutions. \(\Box\)

Now we return to the proof of (6.49). First of all we observe that (6.34) and (6.56) imply that

\[
 \overline{A}_1 \circ PA_2 \cap \Delta D_4 = \emptyset. \quad (6.66)
\]

Hence, the lhs of (6.49) is a subset of its rhs. This is because

\[
 \overline{D}_4 = D_4 \cup \overline{D}_5 \quad \text{and} \quad \overline{D}_5^\# = \overline{D}_5. \quad (6.67)
\]

The first equality follows by applying Lemma 5.2 twice to Lemma 6.1 (4) while the second is covered by Lemma 6.2. To show rhs of (6.49) is a subset of its lhs, it suffices to show that

\[
 \{(\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \overline{A}_1 \circ PA_2 : \pi_2^* \Psi_{PD_4}(\tilde{f}, \tilde{p}, l_{\tilde{p}}) = 0\} \supset \Delta D_5^\#. \quad (6.68)
\]
Claim 6.15. Let \((\bar{f}, \bar{p}, l_\bar{p}) \in \Delta \hat{D}_5^{3÷3} \). Then there exists a solution
\[
(\bar{f}(t_1, t_2), \bar{p}(t_1, t_2), l_{\bar{p}(t_1)}) \in (\mathcal{D} \times \mathbb{P}^2) \cap \mathcal{P} \hat{A}_2
\]

near \((\bar{f}, \bar{p}, l_\bar{p})\) to the set of equations
\[
\begin{align*}
\pi_1^* \psi_{\mathcal{A}_0}(\bar{f}(t_1, t_2), \bar{p}(t_1, t_2), l_{\bar{p}(t_1)}) &= 0, \\
\pi_1^* \psi_{\mathcal{A}_1}(\bar{f}(t_1, t_2), \bar{p}(t_1, t_2), l_{\bar{p}(t_1)}) &= 0, \\
\bar{p}(t_1, t_2) &\neq \bar{p}(t_1).
\end{align*}
\]

(6.69)

It is easy to see that Claim 6.15 implies (6.68).

Proof. Choose homogeneous coordinates \([X : Y : Z]\) so that \(\bar{p} = [0 : 0 : 1]\) and let \(\mathcal{U}_p, \pi_x, \pi_y, v_1, w, v, \eta_t, x_t, y_t, t_2, y_t, f_{ij}(t_1, t_2), F, F_{x t_2}\) and \(F_{y t_2}\) be exactly the same as defined in the proof of Claim 6.12. Since \((\bar{f}(t_1, t_2), l_{\bar{p}(t_1)}) \in \mathcal{P} \hat{A}_2\), we conclude that
\[
f_{00}(t_1, t_2) = f_{10}(t_1, t_2) = f_{01}(t_1, t_2) = f_{20}(t_1, t_2) = f_{11}(t_1, t_2) = 0.
\]

The functional equation (6.69) has a solution if and only if the following set of equations has a solution (as numbers):
\[
F = 0, \quad F_{x t_2} = 0, \quad F_{y t_2} = 0, \quad (x_t, y_t) \neq (0, 0) \quad \text{(but small)}.
\]

(6.70)

For the convenience of the reader, let us rewrite the expression for \(F\):
\[
F := \frac{f_{02}(t_1, t_2)}{2} y_t^4 + \frac{f_{30}(t_1, t_2)}{6} x_t^3 + \frac{f_{21}(t_1, t_2)}{2} x_t^2 y_t + \frac{f_{12}(t_1, t_2)}{2} x_t y_t^2 + \frac{f_{03}(t_1, t_2)}{6} y_t^3 + \cdots.
\]

Let us define
\[
\beta_1 := f_{21}^2 - f_{12} f_{30}, \quad \beta_2^\pm := -\frac{f_{03}}{12} - \frac{f_{21}^3}{6 f_{30}^2} + \frac{f_{12} f_{21}}{4 f_{30}} \pm \sqrt{\beta_1 \left( \frac{f_{21}^2}{6 f_{30}^2} - \frac{f_{12}}{6 f_{30}} \right)} \quad \text{and} \\
\beta_3 := f_{30} f_{03}^2 - 6 f_{03} f_{12} f_{21} f_{30} + 4 f_{12}^3 f_{30} + 4 f_{03} f_{21}^3 - 3 f_{12} f_{21}^2 = 4 f_{30}^2 \beta_2^+ \beta_2^-.
\]

(6.71)

Define \(\beta_k(t_1, t_2)\) similarly with \(f_{ij}\) replaced by \(f_{ij}(t_1, t_2)\). Since \((\bar{f}, \bar{p}, l_\bar{p}) \in \hat{D}_5\), the cubic
\[
\Phi(\theta) := \frac{f_{30}}{6} \theta^3 + \frac{f_{21}}{2} \theta^2 + \frac{f_{12}}{2} \theta + \frac{f_{03}}{6}
\]

has a repeated root, but not all the three roots are the same. Hence we conclude that
\[
\beta_3 = 0, \quad \beta_1 \neq 0 \quad \text{and} \quad f_{30} \neq 0.
\]

(6.72)
The last inequality follows from the fact that \((\tilde{f}, \tilde{p}, l\tilde{p})\) belongs to \(\hat{D}^\#_5\) as opposed to \(\hat{D}_5\). We will now construct solutions to (6.70). Corresponding to each branch of \(\sqrt{\beta_1(t_1, t_2)}\), the solutions are:

\[
x_{t_2} = \frac{-f_{21}(t_1, t_2) + \sqrt{\beta_1(t_1, t_2)}}{f_{30}(t_1, t_2)} y_{t_2} + O(y_{t_2}^2),
\]

\[
f_{02}(t_1, t_2) = O(y_{t_2}), \quad \beta_2^+(t_1, t_2) = O(y_{t_2}). \tag{6.73}
\]

Let us explain how we obtained these solutions. To obtain the value of \(x_{t_2}\) we used \(F_{x_{t_2}} = 0\). To obtain the value of \(f_{02}(t_1, t_2)\) we used \(F_{y_{t_2}} = 0\) and the value of \(x_{t_2}\) from the previous equation. Finally we used the fact that \(2F - y_{t_2} F_{y_{t_2}} = 0\) and the value of \(x_{t_2}\) to obtain \(\beta_2^+(t_1, t_2)\). We get a similar solution for the other branch of \(\sqrt{\beta_1(t_1, t_2)}\).

By (6.71) and (6.73), we conclude that as \(y_{t_2}\) goes to zero, \(f_{02}(t_1, t_2)\) and \(\beta_3\) go to zero. Hence, the solutions in (6.73) lie in \(\overline{A}_1 \circ \overline{P}A_2\) and converge to a point \((\tilde{f}, \tilde{p}, l\tilde{p})\) in \(\hat{D}^\#_5\). \(\Box\)

This completes the proof of Lemma 6.3 (3). \(\Box\)

**Proof of Lemma 6.3 (4).** By Lemma 6.11 \((k = 3)\), if we show that

\[
\{(\tilde{f}, \tilde{p}, l\tilde{p}) \in \overline{A}_1 \circ \overline{P}A_3 : \pi^*_2 \Psi_{PD}(\tilde{f}, \tilde{p}, l\tilde{p}) = 0\} = \Delta \overline{PD}^y_5 \cup \Delta \overline{PD}_6 \tag{6.74}
\]

then we have

\[
\{(\tilde{f}, \tilde{p}, l\tilde{p}) \in \overline{A}_1 \circ \overline{P}A_3\} \subseteq \Delta \overline{PA}_5 \cup \Delta \overline{PD}^y_5 \cup \Delta \overline{PD}_6.
\]

By Lemma 6.1 (11), we conclude that

\[
\{(\tilde{f}, \tilde{p}, l\tilde{p}) \in \overline{A}_1 \circ \overline{P}A_3\} \subseteq \Delta \overline{PA}_5 \cup \Delta \overline{PD}^y_5.
\]

On the other hand, by (6.50) applied with \(k = 3\) and (6.74) we conclude that

\[
\{(\tilde{f}, \tilde{p}, l\tilde{p}) \in \overline{A}_1 \circ \overline{P}A_3\} \supseteq \Delta \overline{PA}_5 \cup \Delta \overline{PD}^y_5.
\]

Thus, it suffices to prove (6.74). Note that the intersection of the lhs of (6.74) with \(\hat{D}_3\) is empty. This follows from (6.66) and the fact that \(\overline{PA}_3\) is a subset of \(\overline{PA}_2\) (see Lemma 6.1). Eq. (6.67) and Lemma 6.2 (4) now implies that the lhs of (6.74) is a subset of \(\Delta \overline{PD}^y_5 \cup \Delta \overline{PD}_5\). However, the intersection of the lrhs of (6.74) with \(\Delta PD_5\) is also empty by (6.57). By Lemma 6.1 (6) and Lemma 6.2 (7) we have that

\[
\overline{PD}_5 = PD_5 \cup \overline{PD}_6 \cup \overline{PE}_6 \quad \text{and} \quad \overline{PE}_6 \subset \overline{PD}_5^y.
\]

Hence the lrhs of (6.74) is a subset of its rhs. To show the converse, we will first simultaneously prove the following three statements:
\[ \overline{A}_1 \circ \mathcal{P} A_3 \supset \Delta \mathcal{P} D_5^\gamma , \] (6.75)
\[ \overline{A}_1 \circ \mathcal{P} D_4 \cap \Delta \mathcal{P} D_5^\gamma = \emptyset , \] (6.76)
\[ \overline{A}_1 \circ \mathcal{P} A_4 \cap \Delta \mathcal{P} D_5^\gamma = \emptyset . \] (6.77)

And then we will prove the following two statements simultaneously:
\[ \overline{A}_1 \circ \mathcal{P} A_3 \supset \Delta \mathcal{P} D_6, \] (6.78)
\[ \overline{A}_1 \circ \mathcal{P} A_4 \cap \Delta \mathcal{P} D_6 = \emptyset . \] (6.79)

Note that (6.75) and (6.78) imply rhs of (6.74) is a subset of its lhs, since \( \overline{A}_1 \circ \mathcal{P} A_3 \) is a closed set.

**Claim 6.16.** Let \((\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \Delta \mathcal{P} D_5^\gamma \). Then there exists a solution
\[ (\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) \in (\mathcal{D} \times \mathbb{P}^2) \circ \mathcal{P} A_3 \]

near \((\tilde{f}, \tilde{p}, l_{\tilde{p}})\) to the set of equations
\[ \pi_1^* \psi_{A_0}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) = 0, \]
\[ \pi_1^* \psi_{A_1}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) = 0, \quad \tilde{p}(t_1, t_2) \neq \tilde{p}(t_1). \] (6.80)

Moreover, such solution lies in \(\overline{A}_1 \circ \mathcal{P} A_3\), i.e.,
\[ \pi_2^* \psi_{\mathcal{P} D_1}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) \neq 0, \] (6.81)
\[ \pi_2^* \psi_{\mathcal{P} A_1}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) \neq 0. \] (6.82)

In particular, \((\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)})\) does not lie in \(\overline{A}_1 \circ \mathcal{P} D_4\) or \(\overline{A}_1 \circ \mathcal{P} A_4\).

Note that Claim 6.16 implies (6.75), (6.76) and (6.77) simultaneously.

**Proof.** Choose homogeneous coordinates \([X : Y : Z]\) so that \(\tilde{p} = [0 : 0 : 1]\) and let \(U_{\tilde{p}}, \pi_x, \pi_y, v_1, w, v, \eta, \eta_{t_1}, \eta_{t_2}, x_{t_1}, y_{t_1}, x_{t_2}, y_{t_2}, f_{i j}(t_1, t_2), F, F_{x t_2}\) and \(F_{y t_2}\) be exactly the same as defined in the proof of Claim 6.12, except for one difference: we take \((\tilde{f}(t_1, t_2), l_{\tilde{p}(t_1)})\) to be a point in \(\overline{P} A_3\). Hence
\[ f_{00}(t_1, t_2) = f_{10}(t_1, t_2) = f_{01}(t_1, t_2) = f_{11}(t_1, t_2) = f_{20}(t_1, t_2) = f_{30}(t_1, t_2) = 0. \]

The functional equation (6.80) has a solution if and only if the following set of equations has a solution (as numbers):
\[ F = 0, \quad F_{x t_2} = 0, \quad F_{y t_2} = 0, \quad (x_{t_2}, y_{t_2}) \neq (0, 0) \quad \text{(but small)}. \] (6.83)
For the convenience of the reader, let us rewrite the expression for $F$:

$$F := \frac{f_{02}(t_1, t_2)}{2} y_{t_2}^2 + \frac{f_{21}(t_1, t_2)}{2} x_{t_2}^2 y_{t_2} + \frac{f_{12}(t_1, t_2)}{2} x_{t_2} x_{y_{t_2}} + \frac{f_{03}(t_1, t_2)}{6} y_{t_2}^3 + \cdots.$$ 

Since $(\tilde{f}, \tilde{p}, l_p) \in \Delta \mathcal{PD}_Y^\gamma$, we claim that

$$f_{21} \neq 0, \quad \beta_1 := \frac{(f_{12} \partial_x - 2f_{21} \partial_y)^3 f}{2f_{21}^2} = 3f_{12}^2 - 4f_{21}f_{03} = 0 \quad \text{and} \quad (6.84)$$

$$\beta_2 := (f_{12} \partial_x - 2f_{21} \partial_y)^4 f = f_{12}^4 f_{40} - 8f_{12}^3 f_{21} f_{31} + 24f_{12}^2 f_{21} f_{22} - 32f_{12} f_{21}^2 f_{13} + 16f_{21}^4 \neq 0. \quad (6.85)$$

Let us justify this. Since $(\tilde{f}, \tilde{p}) \in \mathcal{D}_5$ there exists a non-zero vector $u = m_1 v + m_2 w$ such that

$$\nabla^3 f|_{\tilde{p}}(u, u, v) = m_1^2 f_{30} + 2m_1 m_2 f_{21} + m_2^2 f_{12} = 0, \quad (6.86)$$

$$\nabla^3 f|_{\tilde{p}}(u, u, w) = m_1^2 f_{21} + 2m_1 m_2 f_{12} + m_2^2 f_{03} = 0, \quad (6.87)$$

$$\nabla^4 f|_{\tilde{p}}(u, u, u, v) \neq 0. \quad (6.88)$$

Since $(\tilde{f}, l_p) \in \mathcal{PD}_Y^\gamma$, we conclude by definition that

$$f_{30} = 0, \quad f_{21} \neq 0, \quad m_2 \neq 0. \quad (6.89)$$

(If $m_2 = 0$ then $f_{21}$ would be zero.) Eqs. (6.89) and (6.86) now imply that

$$m_1/m_2 = -f_{12}/(2f_{21}). \quad (6.90)$$

Eqs. (6.90) and (6.87) imply (6.84). Finally, (6.88) implies (6.85).

We claim that solutions to (6.83) are given by

$$x_{t_2} = -\frac{f_{12}(t_1, t_2)}{2f_{21}(t_1, t_2)} y_{t_2} + O(y_{t_2}^2), \quad \beta_1(t_1, t_2) = -\frac{\beta_2(t_1, t_2)}{8f_{21}(t_1, t_2)^3} y_{t_2} + O(y_{t_2}^2) \quad \text{and}$$

$$f_{02}(t_1, t_2) = -\frac{\beta_2(t_1, t_2)}{192f_{21}(t_1, t_2)^4} y_{t_2}^2 + O(y_{t_2}^3). \quad (6.91)$$

Let us explain how we obtained these solutions. Assuming $y_{t_2} \neq 0$ (to be justified at the end) define $L := \frac{x_{t_2}}{y_{t_2}}$. Using $x_{t_2} = L y_{t_2}$ with $y_{t_2} \neq 0$ in the equation $F_{x_{t_2}} = 0$ we can solve for $L$ via the Implicit Function Theorem. That gives us

$$L = -\frac{f_{12}(t_1, t_2)}{2f_{21}(t_1, t_2)} + \left(-\frac{f_{13}(t_1, t_2)}{6f_{21}(t_1, t_2)} + \frac{f_{12}(t_1, t_2)}{4f_{21}(t_1, t_2)^2} f_{22}(t_1, t_2) - \frac{f_{12}(t_1, t_2)^2 f_{31}(t_1, t_2)}{8f_{21}(t_1, t_2)^3} + \frac{f_{12}(t_1, t_2)^3 f_{40}(t_1, t_2)}{48f_{21}(t_1, t_2)^4} \right)y_{t_2} + O(y_{t_2}^2). \quad (6.92)$$
Next, using the equation $2F - y_{t_2} F_{y_{t_2}} = 0$ and the fact that $x_{t_2} = Ly_{t_2}$ and (6.92), we obtain the expression for $\beta_1(t_1, t_2)$ in (6.91). Next, observe that

$$f_{30}(t_1, t_2) = \frac{3f_{12}(t_1, t_2)^2 - \beta_1(t_1, t_2)}{4f_{21}(t_1, t_2)}.$$  \hfill (6.93)

Finally, using the equation $F_{y_{t_2}} = 0$, the fact that $x_{t_2} = Ly_{t_2}$, (6.92), the expression for $\beta_1(t_1, t_2)$ in (6.91) and (6.93) we obtain the expression for $f_{02}(t_1, t_2)$ in (6.91). It is now easy to see that (6.81) holds. It remains to show that $y_{t_2} \neq 0$. To see why that is so, suppose $y_{t_2} = 0$. Then using the fact that $F_{y_{t_2}} = 0$, we conclude that $f_{21}(t_1, t_2) = O(x_{t_2})$. Hence $f_{21}(t_1, t_2)$ goes to zero as $x_{t_2}$ goes to zero, contradicting (6.84).

Finally, (6.82) is true because the section $\pi_2^*\Psi_{P,A_4}$ does not vanish on $\Delta PD_5'$.

\textbf{Corollary 6.17.} Let $\mathbb{V} \to D \times \mathbb{P}^2 \times \mathbb{P}TP^2$ be a vector bundle such that the rank of $\mathbb{V}$ is same as the dimension of $\Delta PD_5'$ and $Q : D \times \mathbb{P}^2 \times \mathbb{P}TP^2 \to \mathbb{V}$ a generic smooth section. Suppose $(\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \Delta PD_5' \cap Q^{-1}(0)$. Then the section

$$\pi_2^*\Psi_{PD_4} \oplus Q : \overline{A_1} \circ P \overline{A_3} \to \pi_2^*(L_{PD_4}) \oplus \mathbb{V}$$

vanishes around $(\tilde{f}, \tilde{p}, l_{\tilde{p}})$ with a multiplicity of 2.

\textbf{Proof.} This follows by observing that at $\Delta PD_5'$, the sections induced by $f_{02}$ and $\beta_1$ (the corresponding functionals) are transverse to the zero set over $P \overline{A_3}$, \textsuperscript{14} $Q$ is generic and (6.91).

\textbf{Claim 6.18.} Let $(\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \Delta PD_6$. Then there exist solutions

$$(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) \in (D \times \mathbb{P}^2) \circ P \overline{A_3}$$

near $(\tilde{f}, \tilde{p}, l_{\tilde{p}})$ to the set of equations

\begin{align*}
\pi_1^*\psi_{A_4}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) &= 0, & \pi_1^*\psi_{A_1}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) &= 0, \\
\pi_2^*\Psi_{PD_4}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) &\neq 0, & \tilde{p}(t_1, t_2) &\neq \tilde{p}(t_1). \hfill (6.94)
\end{align*}

Moreover, any such solution sufficiently close to $(\tilde{f}, \tilde{p}, l_{\tilde{p}})$ lies in $\overline{A_1} \circ P \overline{A_3}$, i.e.,

$$\pi_2^*\Psi_{PD_4}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) \neq 0. \hfill (6.95)$$

In particular, $(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)})$ does not lie in $\overline{A_1} \circ P \overline{A_4}$.

\textsuperscript{14} To see why; just take the partial derivative with respect to $f_{02}$ and $f_{03}$. Since $f_{21} \neq 0$, transversality follows.
It is easy to see that Claim 6.18 implies (6.78) and (6.79) simultaneously.

**Proof.** Choose homogeneous coordinates \([X : Y : Z]\) so that \(\tilde{p} = [0 : 0 : 1]\) and let \(U_\rho, \pi_x, \pi_y, v_1, w, v, n, x_t, y_t, f_{ij}(t_1, t_2), F, F_{x_t}\) and \(F_{y_t}\) be exactly the same as defined in the proof of Claim 6.16. Since \((\tilde{f}(t_1, t_2), l_{\tilde{p}(t_1)}) \in \Delta \mathcal{P} \mathcal{A}_3\) we conclude that

\[
f_{00}(t_1, t_2) = f_{10}(t_1, t_2) = f_{01}(t_1, t_2) = f_{11}(t_1, t_2) = f_{20}(t_1, t_2) = f_{30}(t_1, t_2) = 0.
\]

Furthermore, since \((\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \Delta \mathcal{P} \mathcal{D}_6\), we conclude that

\[
f_{21}, f_{40} = 0 \quad \text{and} \quad f_{12}, \mathcal{D}_7^f \neq 0. \quad (6.96)
\]

The functional equation (6.94) has a solution if and only if the following has a numerical solution:

\[
F = 0, \quad F_{x_t} = 0, \quad F_{y_t} = 0, \quad (x_{t_2}, y_{t_2}) \neq (0, 0) \quad \text{(but small)}. \quad (6.97)
\]

For the convenience of the reader, let us rewrite the expression for \(F\):

\[
F := \frac{f_{02}(t_1, t_2)}{2} y_{t_2}^2 + \frac{f_{21}(t_1, t_2)}{2} x_{t_2} y_{t_2} + \frac{f_{12}(t_1, t_2)}{2} x_{t_2} y_{t_2}^2 + \frac{f_{03}(t_1, t_2)}{6} y_{t_2}^3 + \cdots.
\]

We will now construct solutions to (6.97). Let us define \(G := F - \frac{y_{t_2} F_{y_t}}{2} - \frac{x_{t_2} F_{x_t}}{4}\). Then

\[
G := -\frac{f_{03}(t_1, t_2)}{12} y_{t_2}^3 + y_{t_2}^3 E_1(y_{t_2})
\]

\[
- x_{t_2} \left( \frac{f_{12}(t_1, t_2)}{8} y_{t_2}^2 + \frac{f_{31}(t_1, t_2)}{24} x_{t_2} y_{t_2}^2 + \frac{f_{50}(t_1, t_2)}{480} x_{t_2}^4 + \frac{y_{t_2}^2 E_2(x_{t_2}, y_{t_2})}{y_{t_2}} \right)
\]

\[
+ x_{t_2} y_{t_2} E_3(x_{t_2}) + x_{t_2} E_4(x_{t_2}),
\]

where \(E_t\) is a holomorphic function vanishing at the origin.

We now make a change of variables by using a holomorphic function \(C(y_{t_2})\) such that by making the substitution \(x_{t_2} = \hat{x}_{t_2} + C(y_{t_2})\), the coefficients of \(y_{t_2}^n\) are killed for all \(n\) in \(G\). We may now make a change of coordinates \(y_{t_2} = \hat{y}_{t_2} + B(\hat{x}_{t_2})\) so that the coefficient of \(\hat{x}_{t_2}^n \hat{y}_{t_2}\) is killed for all \(n\) in \(G\). The existence of these functions follow from an identical argument as in [2], where we give a necessary and sufficient criteria for a curve to have a \(D_k\)-node. After these changes, \(G\) is given by

\[
G = -\frac{f_{12}(t_1, t_2)}{8} \hat{x}_{t_2} \left( \hat{y}_{t_2}^2 (1 + E_5(\hat{x}_{t_2}, \hat{y}_{t_2})) + \frac{\mathcal{D}_7^f(t_1, t_2)}{60 f_{12}(t_1, t_2)} \hat{x}_{t_2}^4 + O(\hat{x}_{t_2}^5) \right).
\]

Here \(E_5\) is a holomorphic function which vanishes at the origin. Hence we are solving, in terms of the new variables \(\hat{x}_{t_2}\) and \(\hat{y}_{t_2}\), for
\[ G = 0, \quad F_{\dot{x}_t} = 0, \quad F_{\dot{y}_t} = 0. \] (6.98)

Note that \( F_{\dot{x}_t} \) and \( F_{\dot{y}_t} \) are partials with respect to \( x_{t_2} \) and \( y_{t_2} \) expressed in the new coordinates, they are not the partials with respect to \( \dot{x}_{t_2} \) and \( \dot{y}_{t_2} \). Let us write \( C(y_{t_2}) \) and \( B(\dot{x}_{t_2}) \) to second order:

\[
C(y_{t_2}) = -2f_{03}(t_1, t_2)g_{12}(t_1, t_2) + \left( -f_{04}(t_1, t_2) + \frac{2f_{03}(t_1, t_2)f_{31}(t_1, t_2)}{3f_{12}(t_1, t_2)^2} \right) y_{t_2} + O(y_{t_2}^3),
\]

\[
B(\dot{x}_{t_2}) = -\frac{f_{31}(t_1, t_2)}{6f_{12}(t_1, t_2)}\dot{x}_{t_2}^2 + b_3\dot{x}_{t_2}^3 + O(\dot{x}_{t_2}^4).
\]

The solutions to (6.98) are given by

\[
\dot{y}_{t_2} = \alpha^{2}\dot{x}_{t_2}^2 + p_3\dot{x}_{t_2}^3 + O(\dot{x}_{t_2}^4), \quad (6.99)
\]

\[
f_{21}(t_1, t_2) = \left( -2\alpha f_{02}(t_1, t_2) + \frac{f_{02}(t_1, t_2)f_{31}(t_1, t_2)}{3f_{12}(t_1, t_2)} \right)
+ \left[ -2(b_3 + p_3)f_{02}(t_1, t_2) - \frac{8\alpha^2 f_{02}(t_1, t_2)f_{03}(t_1, t_2)}{3f_{12}(t_1, t_2)} - 2\alpha f_{12}(t_1, t_2) \right] \dot{x}_{t_2}
+ O(\dot{x}_{t_2}^2) \quad (6.100)
\]

\[
f_{40}(t_1, t_2) = \left( 12\alpha^2 f_{02}(t_1, t_2) - \frac{4\alpha f_{02}(t_1, t_2)f_{31}(t_1, t_2)}{f_{12}(t_1, t_2)} + \frac{f_{02}(t_1, t_2)f_{31}(t_1, t_2)^2}{3f_{12}(t_1, t_2)^2} \right)
+ \left( 24\alpha(b_3 + p_3)f_{02}(t_1, t_2) + \frac{32\alpha^3 f_{02}(t_1, t_2)f_{03}(t_1, t_2)}{f_{12}(t_1, t_2)} \right.
\]
\[
+ 24\alpha^2 f_{12}(t_1, t_2) - 4\alpha f_{31}(t_1, t_2)
\]
\[
- \frac{16\alpha^2 f_{02}(t_1, t_2)f_{03}(t_1, t_2)f_{31}(t_1, t_2)}{f_{12}(t_1, t_2)^2}
+ \frac{4(b_3 + p_3)f_{02}(t_1, t_2)f_{31}(t_1, t_2)}{f_{12}(t_1, t_2)}
\]
\[
+ 8\alpha f_{02}(t_1, t_2)f_{03}(t_1, t_2)f_{31}(t_1, t_2)^2
\]
\[
- \frac{4f_{02}(t_1, t_2)f_{03}(t_1, t_2)f_{31}(t_1, t_2)^3}{27f_{12}(t_1, t_2)^4} \right) \dot{x}_{t_2} + O(\dot{x}_{t_2}^2),
\]

where \( \alpha := \sqrt{-\frac{D(t_1, t_2)}{60f_{12}(t_1, t_2)}} \) is a branch of the square root. (6.101)

The exact values of \( b_3 \) and \( p_3 \) are not important, but they play a role in the calculation. Note that each value of \( \alpha \) corresponds to a different solution. Let us now explain how
we obtained these solutions. First of all we claim that \( \hat{x}_{t_2} \neq 0 \); we will justify that at the end. Assuming that, we obtain (6.99) from the fact \( G = 0 \). Next, we obtain (6.100) from (6.99) and using the fact that \( F_{y_1} = 0 \). Finally, we obtain (6.101) from (6.99), (6.100) and using the fact that \( F_{x_1} = 0 \). Observe that Eqs. (6.99), (6.100) and (6.101) imply that

\[
 f_{02}(t_1, t_2) A_4^{f(t_1, t_2)} = \frac{1}{5} D_7^{f(t_1, t_2)} f_{12}(t_1, t_2) \hat{x}_{t_2}^2 + \hat{x}_{t_2}^2 E_6(\hat{x}_{t_2}, f_{02}(t_1, t_2))
\]

where \( E_6(0, 0) = 0 \). Hence, if \( f_{02}(t_1, t_2) \) and \( \hat{x}_{t_2} \) are small and non-zero, \( f_{02}(t_1, t_2) A_4^{f(t_1, t_2)} \) is non-zero. Hence (6.95) holds.

It remains to show that \( \hat{x}_{t_2} \neq 0 \). If \( \hat{x}_{t_2} = 0 \), then using the fact that \( F_{x_1} = 0 \) we get

\[
 f_{12}(t_1, t_2) = \frac{4 f_{03}(t_1, t_2) f_{21}(t_1, t_2)}{3 f_{12}(t_1, t_2)} + O(\hat{y}_{t_2}).
\]

Hence \( f_{12}(t_1, t_2) \) goes to zero as \( f_{21}(t_1, t_2) \) and \( \hat{y}_{t_2} \) go to zero, which contradicts (6.96). \( \square \)

This proves Lemma 6.3 (4). \( \square \)

Before proceeding further, note that (6.57) and (6.79) imply that

\[
 \Delta PD_7^4 \subset \Delta PD_7. \tag{6.103}
\]

**Proof of Lemma 6.3 (5).** By Lemma 6.11 and (6.50) for \( k = 4 \), it suffices to show that

\[
 \{(f, \hat{p}, l_p) \in \overline{A}_1 \circ PD_4 : \pi_2 \Psi_{PD_4}(f, \hat{p}, l_p) = 0 \} = \Delta PD_7^4 \cup \Delta PE_6. \tag{6.104}
\]

By the definition of \( \Delta PD_7^4 \), to prove (6.104) it suffices to show that

\[
 \{(f, \hat{p}, l_p) \in \overline{A}_1 \circ PD_4 : \pi_2 \Psi_{PD_4}(f, \hat{p}, l_p) = 0, \pi_2 \Psi_{PE_6}(f, \hat{p}, l_p) = 0 \}
 = \Delta PE_6. \tag{6.105}
\]

It is clear that the lhs of (6.105) is a subset of its rhs. To prove the converse, let us prove the following three facts simultaneously:

\[
 \overline{A}_1 \circ PA_4 \subset \Delta PE_6, \tag{6.106}
\]

\[
 \overline{A}_1 \circ PA_5 \cap \Delta PE_6 = \emptyset, \tag{6.107}
\]

\[
 \overline{A}_1 \circ PD_4 \cap \Delta PE_6 = \emptyset. \tag{6.108}
\]

Note that since \( \overline{A}_1 \circ PA_4 \) is a closed set, (6.106) implies that the rhs of (6.105) is a subset of its lhs. We will need (6.108) later; since it follows from the present setup, we prove it here.
Claim 6.19. Let \((\bar{f}, \bar{p}, l_p) \in \Delta \mathcal{P} \mathcal{E}_6\). Then there exist solutions
\[
(\bar{f}(t_1, t_2), \bar{p}(t_1, t_2), l_{\bar{p}(t_1)}) \in (\bar{D} \times \mathbb{P}^2) \circ \bar{\mathcal{P}} \mathcal{A}_3
\]

near \((\bar{f}, \bar{p}, l_p)\) to the set of equations
\[
\pi_1^* \psi_{\mathcal{A}_0}(\bar{f}(t_1, t_2), \bar{p}(t_1, t_2), l_{\bar{p}(t_1)}) = 0, \quad \pi_1^* \psi_{\mathcal{A}_1}(\bar{f}(t_1, t_2), \bar{p}(t_1, t_2), l_{\bar{p}(t_1)}) = 0, \\
\pi_2^* \psi_{\mathcal{P} \mathcal{A}_4}(\bar{f}(t_1, t_2), \bar{p}(t_1, t_2), l_{\bar{p}(t_1)}) = 0, \quad \bar{p}(t_1, t_2) \neq \bar{p}(t_1). \quad (6.109)
\]

Moreover, any such solution sufficiently close to \((\bar{f}, \bar{p}, l_p)\) lies in \(\mathcal{A}_1 \circ \mathcal{P} \mathcal{A}_4\), i.e.,
\[
\pi_2^* \psi_{\mathcal{P} \mathcal{A}_5}(\bar{f}(t_1, t_2), \bar{p}(t_1, t_2), l_{\bar{p}(t_1)}) \neq 0, \quad (6.110) \\
\pi_2^* \psi_{\mathcal{P} \mathcal{D}_4}(\bar{f}(t_1, t_2), \bar{p}(t_1, t_2), l_{\bar{p}(t_1)}) \neq 0. \quad (6.111)
\]

In particular, \((\bar{f}(t_1, t_2), \bar{p}(t_1, t_2), l_{\bar{p}(t_1)})\) does not lie in \(\overline{\mathcal{A}_1 \circ \mathcal{P} \mathcal{A}_5}\) or \(\overline{\mathcal{A}_1 \circ \mathcal{P} \mathcal{D}_4}\).

It is easy to see that Claim 6.19 implies (6.106), (6.107) and (6.108) simultaneously.

Proof. Choose homogeneous coordinates \([X : Y : Z]\) so that \(\bar{p} = [0 : 0 : 1]\) and let \(\mathcal{U}_{\bar{p}}, \pi_X, \pi_Y, v_1, w, \eta, \eta_t, x_{t_1}, y_{t_1}, x_{t_2}, y_{t_2}, f_{ij}(t_1, t_2), \mathcal{F}, \mathcal{F}_{x_{t_2}}\) and \(\mathcal{F}_{y_{t_2}}\) be exactly the same as defined in the proof of Claim 6.16. Since \((\bar{f}(t_1, t_2), l_{\bar{p}(t_1)}) \in \overline{\mathcal{P} \mathcal{A}_3}\), we conclude that
\[
f_{00}(t_1, t_2) = f_{10}(t_1, t_2) = f_{01}(t_1, t_2) = f_{11}(t_1, t_2) = f_{20}(t_1, t_2) = f_{30}(t_1, t_2) = 0.
\]

Moreover, since \((\bar{f}, \bar{p}, l_p) \in \Delta \mathcal{P} \mathcal{E}_6\), we conclude that
\[
f_{21}, f_{12} = 0, \quad f_{03}, f_{40} \neq 0. \quad (6.112)
\]

The functional equation (6.109) has a solution if and only if the following set of equations has a solution (as numbers):
\[
\mathcal{F} = 0, \quad \mathcal{F}_{x_{t_2}} = 0, \quad \mathcal{F}_{y_{t_2}} = 0, \\
f_{02}(t_1, t_2)\mathcal{A}_4^{f(t_1, t_2)} = 0, \quad (x_{t_2}, y_{t_2}) \neq (0, 0) \quad \text{(small)}. \quad (6.113)
\]

Since \(f_{02}(t_1, t_2)\mathcal{A}_4^{f(t_1, t_2)} = 0\) we conclude that \(f_{02}(t_1, t_2) = \frac{3f_{21}(t_1, t_2)^2}{f_{40}(t_1, t_2)}\). Hence
\[
\mathcal{F} = \frac{3f_{21}(t_1, t_2)^2}{2f_{40}(t_1, t_2)}y_2 + \frac{f_{21}(t_1, t_2)}{2}x_{t_2}y_{t_2} + \frac{f_{12}(t_1, t_2)}{2}x_{t_2}y_{t_2} + \frac{f_{03}(t_1, t_2)}{6}y_{t_2}^3 \\
+ \frac{f_{40}(t_1, t_2)}{24}x_{t_2}^4 + \frac{\mathcal{R}_{50}(x_{t_2})}{120}x_{t_2}^5 + \frac{\mathcal{R}_{31}(x_{t_2})}{6}x_{t_2}^3y_{t_2} + \frac{\mathcal{R}_{22}(x_{t_2})}{4}x_{t_2}^2y_{t_2}^2 \\
+ \frac{\mathcal{R}_{13}(x_{t_2}, y_{t_2})}{4}x_{t_2}y_{t_2}^2 + \frac{\mathcal{R}_{04}(y_{t_2})}{24}y_{t_2}^4.
\]

We will now eliminate $f_{12}(t_1,t_2)$ and $f_{21}(t_1,t_2)$ from (6.113). First we make a simple observation: let

$$A(\theta) := A_0 + A_1 \theta + A_2 \theta^2, \quad B(\theta) := B_0 + B_1 \theta,$$

$$p_1 := -A_2 B_1, \quad p_2 := -A_2^2 B_0 + A_1 A_2 B_1 + A_2^2 B_1 \theta.$$

Then $p_1 A(\theta) + p_2 B(\theta)$ is independent of $\theta$. With this observation we will now proceed to define

$$G_1 := F - x_{t_2} F_{x_{t_2}}, \quad G_2 := F - \frac{y_{t_2} F_{y_{t_2}}}{2}, \quad G := P_1(t_1, t_2) G_1 + P_2(t_1, t_2) G_2$$

where

$$P_1 := -\frac{3x^4 y^4}{32 f_{40}},$$

$$P_2 := \frac{3y^7 f_{03}}{16 f_{40}^2} + \frac{9x^2 y^5 f_{21}}{16 f_{40}^2} - \frac{9x^4 y^4}{32 f_{40}} + \frac{3y^8 R_{04}(y)}{32 f_{40}^2} + \frac{3x y^7 R_{13}(x, y)}{16 f_{40}^3} - \frac{3x^3 y^5 R_{31}(x)}{16 f_{40}^2} - \frac{3x^5 y^4 R_{50}(x)}{160 f_{40}^2} + \frac{3y^9 R_{04}^{(1)}(y)}{64 f_{40}^2} + \frac{3x y^8 R_{13}^{(1)}(x, y)}{16 f_{40}^3}.$$ 

The quantity $P_k(t_1, t_2)$ is similarly defined, with $f_{ij}$, $x$ and $y$ replaced by $f_{ij}(t_1, t_2)$, $x_{t_2}$ and $y_{t_2}$.

Note that $G_1$ and $G_2$ are independent of $f_{12}(t_1, t_2)$. Secondly, they are quadratic and linear in $f_{21}(t_1, t_2)$ respectively. Hence, using our previous observation, $G$ is independent of $f_{12}(t_1, t_2)$ and $f_{21}(t_1, t_2)$.

We claim that $x_{t_2} \neq 0$ and $y_{t_2} \neq 0$; we will justify that at the end. Assuming this claim we conclude that solving (6.113) is equivalent to solving:

$$G = 0, \quad G_2 = 0, \quad F = 0, \quad (x_{t_2}, y_{t_2}) \neq (0, 0) \text{ (small)}.$$

Define $L := x_{t_2}^4 / y_{t_2}^3$. We will first solve for $L$ in terms of $x_{t_2}$ and $y_{t_2}$ and then we will parametrize $(x_{t_2}, y_{t_2})$. Notice that we can rewrite $G$ in terms of $x_{t_2}$, $y_{t_2}$ and $L$ in such a way that the highest power of $x_{t_2}$ is 3; whenever there is an $x_{t_2}^4$ we replace it with $L y_{t_2}^3$. Hence

$$G = y_{t_2}^{10} \left(-\frac{f_{03}(t_1, t_2)^2}{64 f_{40}(t_1, t_2)^2} + \frac{f_{03}(t_1, t_2)}{64 f_{40}(t_1, t_2)} L + E(x_{t_2}, y_{t_2}, L)\right)$$

where $E(0, 0, L) = 0$. Hence $G = 0$ and $y_{t_2} \neq 0$ implies that

$$\Phi(x_{t_2}, y_{t_2}, L) := \frac{f_{03}(t_1, t_2)^2}{64 f_{40}(t_1, t_2)^2} + \frac{f_{03}(t_1, t_2)}{64 f_{40}(t_1, t_2)} L + E(x_{t_2}, y_{t_2}, L) = 0.$$

Replace $\theta \to f_{21}(t_1, t_2)$, $A(\theta) \to G_1$, $B(\theta) \to G_2$, $p_1 \to P_1(t_1, t_2)$ and $p_2 \to P_2(t_1, t_2)$.

---

15 Replace $\theta \to f_{21}(t_1, t_2)$, $A(\theta) \to G_1$, $B(\theta) \to G_2$, $p_1 \to P_1(t_1, t_2)$ and $p_2 \to P_2(t_1, t_2)$. 
Hence, by the Implicit Function Theorem we conclude that
\[ L(x_{t_2}, y_{t_2}) = \frac{f_{03}(t_1, t_2)}{f_{40}(t_1, t_2)} + E_2(x_{t_2}, y_{t_2}) \]
where \( E_2(0, 0) \) is zero. Hence \((x_{t_2}, y_{t_2})\) is parametrized by
\[ y_{t_2} = u^4, \quad x_{t_2} = \alpha u^3 + O(u^4) \]
where \( \alpha := \sqrt[4]{\frac{f_{03}(t_1, t_2)}{f_{40}(t_1, t_2)}} \), a branch of the fourth root. Note that just one branch of the fourth root gives all the solutions, choosing another branch does not give us any more solutions.\(^{16}\) We have
\[ f_{02}(t_1, t_2) = \frac{f_{03}(t_1, t_2)}{12} u^4 + O(u^5), \quad (6.114) \]
\[ f_{02}(t_1, t_2)^2 A_5^{f(t_1, t_2)} = -\frac{5f_{03}(t_1, t_2)^2 f_{40}(t_1, t_2)}{18\alpha} u^5 + O(u^6). \quad (6.115) \]
To arrive at these, use \( G_2 = 0 \) to get
\[ f_{21}(t_1, t_2) = \frac{f_{03}(t_1, t_2)}{3\alpha^2} u^2 + O(u^3). \]
We then use the fact that \( f_{02}(t_1, t_2) = \frac{3f_{21}(t_1, t_2)^2}{f_{40}(t_1, t_2)} \) to get (6.114). Finally using \( F = 0 \) we get that
\[ f_{12}(t_1, t_2) = -\frac{2\alpha^3 f_{40}(t_1, t_2)}{3} u + O(u^2). \]
Plugging all this in we get (6.115). Eqs. (6.114) and (6.115) imply that (6.110) and (6.111) hold respectively.

It remains to show that \( x_{t_2} \neq 0 \) and \( y_{t_2} \neq 0 \). If \( y_{t_2} = 0 \) then \( F = 0 \) implies that
\[ f_{40}(t_1, t_2) = -\frac{x_{t_2} R_{50}(x_{t_2})}{5}. \]
As \( x_{t_2} \) goes to zero, \( f_{40}(t_1, t_2) \) goes to zero, contradicting (6.112). Similarly, if \( x_{t_2} = 0 \), then using the fact that \( F - \frac{y_{t_2} F_{y t_2}}{2} = 0 \) we get that
\[ f_{03}(t_1, t_2) = -\frac{2y_{t_2} R_{04}(y_{t_2}) + y_{t_2}^2 R_{04}^{(1)}(y_{t_2})}{4}. \]
As \( y_{t_2} \) goes to zero, \( f_{03}(t_1, t_2) \) goes to zero, contradicting (6.112).

\(^{16}\) Observe that a neighbourhood of the origin for the curve \( x^4 - y^3 = 0 \) is just one copy of \( \mathbb{C} \) parametrized by \( x = u^3 \) and \( y = u^4 \).
Corollary 6.20. Let $\mathcal{W} \to D \times \mathbb{P}^2 \times \mathbb{P}T\mathbb{P}^2$ be a vector bundle such that the rank of $\mathcal{W}$ is same as the dimension of $\Delta \mathcal{P}E_6$ and $Q: D \times \mathbb{P}^2 \times \mathbb{P}T\mathbb{P}^2 \to \mathcal{W}$ a generic smooth section. Suppose $(\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \Delta \mathcal{P}E_6 \cap Q^{-1}(0)$. Then the section

$$\pi_2^* \Psi_{\mathcal{P}A_5} \oplus Q : \overline{\mathcal{A}_1} \circ \overline{\mathcal{P}A_4} \longrightarrow \pi_2^*(\mathbb{L}_{\mathcal{P}A_5}) \oplus \mathcal{W}$$

vanishes around $(\tilde{f}, \tilde{p}, l_{\tilde{p}})$ with a multiplicity of 5.

Proof. Follows from the fact that the sections induced by $f_{02}$, $f_{21}$ and $f_{12}$ (the corresponding functionals) are transverse to the zero set over $\Delta \mathcal{P}A_3$,\footnote{To see this we take the partial derivatives with respect to $f_{02}$, $f_{21}$ and $f_{12}$.} the fact that $Q$ is generic and (6.115). □

This completes the proof of Lemma 6.3 (5). □

Proof of Lemma 6.3 (6). We have to show that

$$\{(\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \overline{\mathcal{A}_1} \circ \mathcal{P}A_5\} = \Delta \mathcal{P}A_7 \cup \Delta \mathcal{P}D_8 \cup \Delta \mathcal{P}E_7.$$

By Lemma 6.11 and (6.50), it is equivalent to showing that

$$\{(\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \overline{\mathcal{A}_1} \circ \mathcal{P}A_5 : \pi_2^* \Psi_{\mathcal{P}D_4}(\tilde{f}, \tilde{p}, l_{\tilde{p}}) = 0\} = \Delta \mathcal{P}D_8 \cup \Delta \mathcal{P}E_7. \quad (6.116)$$

By the definition of $\Delta \mathcal{P}D_8$, to prove (6.116) it suffices to show that

$$\{(\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \overline{\mathcal{A}_1} \circ \mathcal{P}A_5 : \pi_2^* \Psi_{\mathcal{P}D_4}(\tilde{f}, \tilde{p}, l_{\tilde{p}}) = 0, \pi_2^* \Psi_{\mathcal{P}E_6}(\tilde{f}, \tilde{p}, l_{\tilde{p}}) = 0\} = \Delta \mathcal{P}E_7. \quad (6.117)$$

Before we prove (6.117), we will first prove the following fact:

$$\Delta \mathcal{P}D_8^s \subset \Delta \mathcal{P}D_8. \quad (6.118)$$

Although (6.118) is not required for the proof of Lemma 6.3 (6), it will be needed later. To prove (6.118), it suffices to show that

$$\overline{\mathcal{A}_1} \circ \mathcal{P}A_5 \cap \Delta \mathcal{P}D_7 = \emptyset. \quad (6.119)$$

To see why, note that by (6.79) combined with $\mathcal{P}A_5 \subset \overline{\mathcal{P}A_4}$ we have

$$\overline{\mathcal{A}_1} \circ \mathcal{P}A_5 \cap \Delta \mathcal{P}D_6 = \emptyset.$$

Since $\Delta \mathcal{P}D_8^s \subset \overline{\mathcal{A}_1} \circ \mathcal{P}A_5$, we conclude that (6.119) implies
\[
\Delta \mathcal{PD}_8 \cap (\Delta \mathcal{PD}_6 \cup \Delta \mathcal{PD}_7) = \emptyset.
\]

On the other hand, by Lemma 6.1 (11) we know that \( \Delta \mathcal{PD}_8 \subseteq \Delta \mathcal{PD}_6 \). Lemma 6.2 (6) now proves (6.118) assuming (6.119).

**Claim 6.21.** Let \((\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \Delta \mathcal{PD}_7\). Then there are no solutions

\[
(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) \in (\mathcal{D} \times \mathbb{P}^2) \circ \mathcal{PA}_3
\]

near \((\tilde{f}, \tilde{p}, l_{\tilde{p}})\) to the set of equations

\[
\begin{align*}
\pi_1^* \psi_{A_0}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) &= 0, & \pi_1^* \psi_{A_1}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) &= 0, \\
\pi_2^* \psi_{\mathcal{PA}_4}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) &= 0, & \pi_2^* \psi_{\mathcal{PA}_5}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) &= 0, \\
\pi_1^* \psi_{\mathcal{PD}_4}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) &\neq 0, & \tilde{p}(t_1, t_2) &\neq \tilde{p}(t_1). \quad (6.120)
\end{align*}
\]

In particular, if \((\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)})\) is sufficiently close to \((\tilde{f}, \tilde{p}, l_{\tilde{p}})\), it does not lie in \(\mathcal{A}_1 \circ \mathcal{PA}_5\).

It is easy to see that Claim 6.21 implies (6.119).

**Proof.** Choose homogeneous coordinates \([X : Y : Z]\) so that \(\tilde{p} = [0 : 0 : 1]\) and let \(U_{\tilde{p}}, \pi_x, \pi_y, v_1, w, v, \eta, t_1, t_2, x_t, y_t, f_{ij}(t_1, t_2), F, F_{x_t}, F_{y_t}\) be exactly the same as defined in the proof of Claim 6.16. Since \((\tilde{f}(t_1, t_2), l_{\tilde{p}(t_1)}) \in \Delta \mathcal{PA}_3\) we conclude that

\[
f_{00}(t_1, t_2) = f_{10}(t_1, t_2) = f_{01}(t_1, t_2) = f_{11}(t_1, t_2) = f_{20}(t_1, t_2) = f_{30}(t_1, t_2) = 0.
\]

Furthermore, since \((\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \Delta \mathcal{PD}_7\), we conclude that

\[
f_{21}, f_{40}, D_7^f = 0 \quad \text{and} \quad f_{12}, D_8^f \neq 0. \quad (6.121)
\]

The functional equation (6.120) has a solution if and only if the following set of equations has a solution (as numbers):

\[
F = 0, \quad F_{x_t} = 0, \quad F_{y_t} = 0, \quad A_4^{f(t_1, t_2)} = 0, \quad A_5^{f(t_1, t_2)} = 0, \\
(x_t, y_t) \neq (0, 0), \quad f_{02}(t_1, t_2) \neq 0 \quad \text{(but small)}. \quad (6.122)
\]

For the convenience of the reader, let us rewrite the expression for \(F\):

\[
F := \frac{f_{02}(t_1, t_2)}{2} y_t^2 + \frac{f_{21}(t_1, t_2)}{2} x_t^2 y_t^2 + \frac{f_{40}(t_1, t_2)}{2} x_t^4 + \frac{f_{12}(t_1, t_2)}{2} x_t^2 y_t^2 + \frac{f_{03}(t_1, t_2)}{6} y_t^3 + \cdots. \quad (6.123)
\]
We will now show that solutions to (6.122) cannot exist. We will show that if 
\((f_{ij}(t_1,t_2),x_{t_2},y_{t_2})\) is a sequence converging to \((f,0,0)\) that satisfies (6.122), then 
\(D^f_{ij}(t_1,t_2)\) goes to zero (after passing to a subsequence). That would contradict (6.121). 
First, we observe that any solution to the set of equations \(A^f_{ij}(t_1,t_2) = 0\) and \(A^f_{0}(t_1,t_2) = 0\) is given by 
\[
 f_{21}(t_1,t_2) = \left( \frac{f_{31}(t_1,t_2) + v}{3f_{12}(t_1,t_2)} \right) f_{02}(t_1,t_2), \quad f_{40}(t_1,t_2) = \frac{3f_{21}(t_1,t_2)^2}{f_{02}(t_1,t_2)}, \\
 D^f_{ij}(t_1,t_2) = -\frac{5v^2}{3f_{12}(t_1,t_2)}. 
\] 
(6.124)

Let us choose the \(+v\) solution for \(f_{21}(t_1,t_2)\), a similar argument will go through for the 
\(-v\) solution. Now, using the value of \(f_{40}(t_1,t_2)\) we observe that the first three terms in 
the expression for \(F\) in (6.123) form a perfect square. We will now rewrite the remaining 
part of \(F\) by making a change of coordinates. Using an identical argument that is given in 
[2] (where we give a necessary and sufficient criteria for a curve to have a \(D_k\)-singularity) 
and using (6.124), we can make a change of coordinates 
\[
 x_{t_2} = \hat{x}_{t_2} + G(y_{t_2},v), \quad y_{t_2} = \hat{y}_{t_2} + B(\hat{x}_{t_2},v) 
\]
so that \(F\) is given by 
\[
 F = u \left( \frac{\hat{y}_{t_2} + B(\hat{x}_{t_2},v)}{6f_{12}(t_1,t_2)} (\hat{x}_{t_2} + J)^2 + \frac{v(\hat{x}_{t_2} + J)^2}{6f_{12}(t_1,t_2)} \right)^2 \\
 - \frac{v^2\hat{x}_{t_2}}{72f_{12}(t_1,t_2)} + \frac{f_{12}(t_1,t_2)}{2}\hat{x}_{t_2}\hat{y}_{t_2}(1 + E_1(\hat{x}_{t_2},\hat{y}_{t_2})) + \frac{D^f_{ij}(t_1,t_2)}{720}\hat{x}_{t_2}^6 + R_4(\hat{x}_{t_2})\hat{x}_{t_2}^7, 
\]
where 
\[
 J := G(\hat{y}_{t_2} + B(\hat{x}_{t_2},v),v), \quad u := f_{02}(t_1,t_2), 
\]
\(E_1\) is a holomorphic function that vanishes at the origin and \(R_4\) is a holomorphic function. 
Note that \(B\) is also a function of \(v\) because the coefficients of \(x_{t_2}^6\) may depend on \(f_{02}(t_1,t_2)\), 
which is equal to \(D^f_{ij}(t_1,t_2) + \frac{5f_{31}(t_1,t_2)^2}{3f_{12}(t_1,t_2)}\). Let 
\[
 \hat{y}_{t_2} := \hat{y}_{t_2}(1 + E_1(\hat{x}_{t_2},\hat{y}_{t_2}))^{\frac{1}{2}} \quad \Rightarrow \quad \hat{y}_{t_2} = \hat{y}_{t_2}(1 + E_2(\hat{x}_{t_2},\hat{y}_{t_2})), 
\]
where \(E_2\) is again a holomorphic function that vanishes at the origin. Let us write \(E_2\) in 
the following way (it will soon be clear why we are doing that): 
\[
 E_2(\hat{x}_{t_2},\hat{y}_{t_2}) = \hat{x}_{t_2}R_1(\hat{x}_{t_2}) + \hat{y}_{t_2}R_2(\hat{x}_{t_2},\hat{y}_{t_2}), 
\]
(6.125)
where \(R_i\) is a holomorphic function. Note that \(R_1\) depends only on \(\hat{x}_{t_2}\). Now, define
\[ z_{t_2} := \hat{y}_{t_2} + \frac{\hat{x}_{t_2}^2 v}{6f_{12}(t_1, t_2)}. \]

Since
\[ G(0, v) = 0 \quad \text{and} \quad B(\hat{x}_{t_2}, v) + \frac{f_{31}(t_1, t_2)}{6f_{12}(t_1, t_2)} \hat{x}_{t_2}^2 = O(\hat{x}_{t_2}^3), \]
we conclude that in the new coordinates \((\hat{x}_{t_2}, z_{t_2})\), \(F\) is given by
\[
F = \frac{u}{2} (z_{t_2} + R_3(\hat{x}_{t_2}, v)\hat{x}_{t_2}^3 + E_3(\hat{x}_{t_2}, z_{t_2}, v)z_{t_2})^2
- \frac{v\hat{x}_{t_2}^2 z_{t_2}}{6f_{12}(t_1, t_2)} + \frac{f_{12}(t_1, t_2)}{2} \hat{x}_{t_2} z_{t_2}^2 + \frac{D_8^{f(t_1, t_2)}}{720} \hat{x}_{t_2}^6 + R_4(\hat{x}_{t_2})\hat{x}_{t_2}^7, \tag{6.126}
\]
where \(E_3(0, 0, v) \equiv 0\) and \(R_i\) are holomorphic functions. Hence, (6.122) has solutions if and only if the following set of equations has a solution
\[
F = 0, \quad F_{\hat{x}_{t_2}} = 0, \quad F_{z_{t_2}} = 0,
\]
\[(\hat{x}_{t_2}, z_{t_2}) \neq (0, 0), \quad u \neq 0, \quad v \text{ small.} \tag{6.127}\]

We will now analyze solutions of (6.127). Note that for any solution \(\hat{x}_{t_2} \neq 0\); if \(\hat{x}_{t_2} = 0\) then \(z_{t_2} \neq 0\) and using (6.126) and \(u \neq 0\) we conclude that \(F \neq 0\), a contradiction. Notice that we can rewrite (6.127) in the following way
\[
p_0 + p_1 w + p_2 v = 0, \quad q_0 + q_1 w + q_2 v = 0, \quad r_0 + r_1 w + r_2 v = 0, \tag{6.128}
\]
where
\[
w := u(z_{t_2} + R_3(\hat{x}_{t_2}, v)\hat{x}_{t_2}^3 + E_3(\hat{x}_{t_2}, z_{t_2}, v)z_{t_2}),
\]
\[
p_0 := \frac{f_{12}(t_1, t_2)}{2} \hat{x}_{t_2} z_{t_2}^2 + \frac{D_8^{f(t_1, t_2)}}{720} \hat{x}_{t_2}^6 + R_4(\hat{x}_{t_2})\hat{x}_{t_2}^7,
\]
\[
p_1 := z_{t_2} + R_3(\hat{x}_{t_2}, v)\hat{x}_{t_2}^3 + E_3(\hat{x}_{t_2}, z_{t_2}, v)z_{t_2},
\]
\[
p_2 := -\frac{\hat{x}_{t_2} z_{t_2}}{6f_{12}(t_1, t_2)},
\]
\[
q_0 := \frac{f_{12}(t_1, t_2)}{2} \hat{x}_{t_2}^2 + \frac{D_8^{f(t_1, t_2)}}{120} \hat{x}_{t_2}^5 + (7R_4(\hat{x}_{t_2}) + \hat{x}_{t_2} R_4\hat{x}_{t_2} (\hat{x}_{t_2}))\hat{x}_{t_2}^6,
\]
\[
q_1 := 3R_3(\hat{x}_{t_2}, v)\hat{x}_{t_2}^2 + R_3(\hat{x}_{t_2}, v)\hat{x}_{t_2}^3 + E_3(\hat{x}_{t_2}, z_{t_2}, v), \quad q_2 := -\frac{\hat{x}_{t_2}^2 z_{t_2}}{2f_{12}(t_1, t_2)},
\]
\[
r_0 := f_{12}(t_1, t_2)\hat{x}_{t_2} z_{t_2}, \quad r_1 := 1 + E_3(\hat{x}_{t_2}, z_{t_2}, v) + z_{t_2} E_3(\hat{x}_{t_2}, z_{t_2}, v),
\]
\[
r_2 := -\frac{\hat{x}_{t_2}^3}{6f_{12}(t_1, t_2)}. \tag{6.129}\]
When a variable appears in a subscript, it means partial derivative with respect to that variable. Since (6.128) holds, we conclude that

$$p_0q_2r_1 - p_0q_1r_2 + p_2q_1r_0 - p_2q_0r_1 + p_1q_0r_2 - p_1q_2r_0 = 0. \tag{6.130}$$

Eqs. (6.130), (6.129) and the fact that $\hat{x}_{t_2} \neq 0$ now imply that

$$\Phi(\hat{x}_{t_2}, z_{t_2}) := z_{t_2}^3 \left(f_{12}(t_1, t_2) + E_4(\hat{x}_{t_2}, z_{t_2}, v)\right) + z_{t_2} \hat{x}_{t_2}^3 \mathcal{R}_5(\hat{x}_{t_2}, z_{t_2}, v) + z_{t_2} \hat{x}_{t_2}^6 \mathcal{R}_6(\hat{x}_{t_2}, z_{t_2}, v) + \hat{x}_{t_2}^5 \mathcal{R}_7(z_{t_2}, \hat{x}_{t_2}, v) = 0, \tag{6.131}$$

where $E_4(0, 0, v) \equiv 0$ and $\mathcal{R}_i(z_{t_2}, \hat{x}_{t_2}, v)$ are holomorphic functions. Now let us define $L := z_{t_2}/\hat{x}_{t_2}$. Let $(f(t_1, t_2), \hat{x}_{t_2}, z_{t_2})$ be a sequence converging to $(f, 0, 0)$ that satisfies (6.127) and such that $(\hat{x}_{t_2}, z_{t_2}) \neq (0, 0)$. It follows from (6.131) that $L$ is bounded, since $f_{12} \neq 0$. Hence, after passing to a subsequence $L$ converges. Since $F = 0$, we can easily see from (6.126) that as $(\hat{x}_{t_2}, z_{t_2}), u$ and $v$ go to zero, $\mathcal{D}_8 f(t_1, t_2)$ goes to zero. This contradicts (6.121).

Now let us prove (6.117). It is clear that the lhs of (6.117) is a subset of its rhs. Next, we will prove the following two facts simultaneously:

$$\overline{A_1 \circ \mathcal{P} A_5} \supset \Delta \mathcal{P} \mathcal{E}_7, \tag{6.132}$$

$$\overline{A_1 \circ \mathcal{P} A_6} \cap \Delta \mathcal{P} \mathcal{E}_7 = \emptyset. \tag{6.133}$$

Since $\overline{A_1 \circ \mathcal{P} A_5}$ is a closed set, (6.132) implies that the rhs of (6.117) is a subset of its lhs.

**Claim 6.22.** Let $(\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \Delta \mathcal{P} \mathcal{E}_7$. Then there exist solutions

$$\left(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}\right) \in \left(\mathcal{D} \times \mathbb{P}^2\right) \circ \mathcal{P} A_3$$

near $(\tilde{f}, \tilde{p}, l_{\tilde{p}})$ to the set of equations

$$\pi_1^* \psi_{A_0}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) = 0, \quad \pi_1^* \psi_{A_1}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) = 0,$$

$$\pi_2^* \psi_{A_4}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) = 0, \quad \pi_2^* \psi_{A_5}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) = 0,$$

$$\pi_2^* \psi_{D_4}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) \neq 0, \quad \tilde{p}(t_1, t_2) \neq \tilde{p}(t_1). \tag{6.134}$$

Moreover, any such solution sufficiently close to $(\tilde{f}, \tilde{p}, l_{\tilde{p}})$ lies in $\overline{A_1 \circ \mathcal{P} A_5}$, i.e.,

$$\pi_2^* \psi_{A_6}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) \neq 0. \tag{6.135}$$

In particular, $(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)})$ does not lie in $\overline{A_1 \circ \mathcal{P} A_6}$.
It is easy to see that Claim 6.22 implies (6.132) and (6.133) simultaneously.

**Proof.** Choose homogeneous coordinates \([X : Y : Z]\) so that \(\tilde{p} = [0 : 0 : 1]\) and let \(\mathcal{U}_{\tilde{p}}, \pi_x, \pi_y, v_1, w, v, \eta, \eta_{t_1}, \eta_{t_2}, x_{t_1}, y_{t_1}, x_{t_2}, y_{t_2}, f_{ij}(t_1, t_2), F, F_{xt_2}, F_{yt_2}\) and \(F_{uy_2}\) be exactly the same as defined in the proof of Claim 6.16. Since \((\tilde{f}(t_1, t_2), l_{\tilde{p}(t_1)}) \in \overline{P\mathcal{A}}_3\), we conclude

\[
f_{00}(t_1, t_2) = f_{10}(t_1, t_2) = f_{01}(t_1, t_2) = f_{11}(t_1, t_2) = f_{20}(t_1, t_2) = f_{30}(t_1, t_2) = 0.
\]

Moreover, since \((\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \Delta P\mathcal{E}_7\), we conclude that

\[
f_{12}, f_{40} = 0, \quad f_{31}, f_{03} \neq 0. \tag{6.136}
\]

The functional equation (6.109) has a solution if and only if the following set of equations has a solution (as numbers):

\[
F = 0, \quad F_{xt_2} = 0, \quad F_{yt_2} = 0, \quad A_4^{f(t_1,t_2)} = 0, \quad A_5^{f(t_1,t_2)} = 0,
\]

\[
f_{02}(t_1, t_2) \neq 0, \quad (x_{t_2}, y_{t_2}) \neq (0, 0) \quad \text{(but small)}. \tag{6.137}
\]

Let us now define the following quantities:

\[
G_1 := F - \frac{x_{t_2}F_{xt_2}}{4} - \frac{y_{t_2}F_{yt_2}}{2}, \quad G_2 := F - \frac{y_{t_2}F_{yt_2}}{2}, \quad G := F + 4G_1 - 2G_2.
\]

Note that \(G_1\) depends linearly on \(f_{12}(t_1, t_2)\) and is independent of \(f_{40}(t_1, t_2), f_{21}(t_1, t_2)\) and \(f_{02}(t_1, t_2)\); \(G_2\) depends linearly on \(f_{40}(t_1, t_2)\) and \(f_{21}(t_1, t_2)\) and is independent of \(f_{12}(t_1, t_2)\) and \(f_{02}(t_1, t_2)\); finally \(G\) depends linearly on \(f_{02}(t_1, t_2)\) and \(f_{40}(t_1, t_2)\) and is independent of \(f_{12}(t_1, t_2)\) and \(f_{21}(t_1, t_2)\). Next, we claim that \(x_{t_2} \neq 0\) and \(y_{t_2} \neq 0\); we will justify that at the end. Assuming this claim, we observe that (6.137) combined with (6.136) is equivalent to

\[
G_1 = 0, \quad G_2 = 0, \quad G = 0,
\]

\[
f_{21}(t_1, t_2) = \frac{5f_{12}(t_1, t_2)f_{40}(t_1, t_2) + f_{02}(t_1, t_2)f_{50}(t_1, t_2)}{10f_{31}(t_1, t_2)},
\]

\[
A_4^{f(t_1,t_2)} = 0, \quad f_{02}(t_1, t_2) \neq 0, \quad x_{t_2} \neq 0, \quad y_{t_2} \neq 0 \quad \text{(but small)}. \tag{6.138}
\]

We will now construct solutions for (6.138). First of all, using \(G = 0\) we can solve for \(f_{02}(t_1, t_2)\) as a function of \(f_{40}(t_1, t_2), x_{t_2}\) and \(y_{t_2}\). Next, using that \(G_1 = 0\), we get \(f_{12}(t_1, t_2)\) as a function of \(x_{t_2}\) and \(y_{t_2}\). Finally, using \(G_2 = 0\), the value of \(f_{02}(t_1, t_2)\), \(f_{12}(t_1, t_2)\) from the previous two equations and the value of \(f_{21}(t_1, t_2)\) from (6.138), we get \(f_{40}(t_1, t_2)\) in terms of \(x_{t_2}\) and \(y_{t_2}\). Plugging the expression back in, we get \(f_{12}(t_1, t_2), f_{21}(t_1, t_2), f_{02}(t_1, t_2)\) and \(f_{40}(t_1, t_2)\) in terms of \(x_{t_2}\) and \(y_{t_2}\).
Next, let us define $L := x_{t_2}^3/y_{t_2}^2$. We note that any expression involving $x_{t_2}$ and $y_{t_2}$ can be re-written in terms of $x_{t_2}, y_{t_2}$ and $L$ so that the highest power of $x_{t_2}$ is 2; replace $x_{t_2}^3$ by $L y_{t_2}^2$. Using that fact, we conclude

$$A_4^{f(t_1, t_2)} \frac{x_{t_2}^4}{y_{t_2}^2} = - \frac{f_{03}(t_1, t_2)^2}{3} + \frac{f_{03}(t_1, t_2)f_{31}(t_1, t_2)}{3} L + E_1(x_{t_2}, y_{t_2}, L) = 0$$

where $E_1(0, 0, L) = 0$. Hence, by the Implicit function theorem, we conclude that

$$L = \frac{f_{03}(t_1, t_2)}{f_{31}(t_1, t_2)} + E_2(x_{t_2}, y_{t_2})$$

where $E_2(0, 0) = 0$. Hence, $x_{t_2}$ and $y_{t_2}$ are parametrized by

$$y_{t_2} = u^3, \quad x_{t_2} = \alpha u^2 + O(u^3)$$

where $\alpha := \sqrt[3]{\frac{f_{03}(t_1, t_2)}{f_{31}(t_1, t_2)}}$, a branch of the cube root.

Plugging in all this we get

$$f_{02}(t_1, t_2) = \frac{f_{03}(t_1, t_2)}{3} u^3 + O(u^4), \quad f_{12}(t_1, t_2) = -\frac{f_{03}(t_1, t_2)}{3\alpha} u + O(u^2), \quad f_{21}(t_1, t_2) = O(u^3), \quad f_{40}(t_1, t_2) = O(u^2),$$

$$f_{02}(t_1, t_2)^3 A_6^{f(t_1, t_2)} = -\frac{10f_{03}(t_1, t_2)^2 f_{31}(t_1, t_2)^2}{9} u^6 + O(u^7). \quad (6.139)$$

Eq. (6.139) implies that (6.135) holds. □

**Corollary 6.23.** Let $\mathbb{W} \to \mathcal{D} \times \mathbb{P}^2 \times \mathbb{P} \mathbb{P}^2$ be a vector bundle such that the rank of $\mathbb{W}$ is same as the dimension of $\Delta \mathcal{P} \mathcal{E}_7$ and $\mathcal{Q} : \mathcal{D} \times \mathbb{P}^2 \times \mathbb{P} \mathbb{P}^2 \to \mathbb{W}$ a generic smooth section. Suppose $(\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \Delta \mathcal{P} \mathcal{E}_7 \cap \mathcal{Q}^{-1}(0)$. Then the section

$$\pi_2^* \mathcal{P} \mathcal{A}_6 \oplus \mathcal{Q} : \overline{\mathcal{A}_1} \circ \mathcal{P} \mathcal{A}_5 \to \pi_2^* (\mathcal{L} \mathcal{P} \mathcal{A}_6) \oplus \mathbb{W}$$

vanishes around $(\tilde{f}, \tilde{p}, l_{\tilde{p}})$ with a multiplicity of 6.

**Proof.** Follows from the fact that the sections induced by $f_{02}, f_{21}, f_{12}$ and $f_{40}$ (the corresponding functionals) are transverse to the zero set over $\Delta \mathcal{P} \mathcal{A}_3$,\(^{18}\) the fact that $\mathcal{Q}$ is generic and (6.139). □

This finishes the proof of Lemma 6.3 (6). □

\(^{18}\) We take partial derivative with respect to $f_{02}, f_{21}, f_{12}$ and $f_{40}$. 
Proof of Lemma 6.3 (7). By definition of $\Delta \mathcal{PD}_6^\dagger$ in (6.1), it suffices to show that
\[
\{(\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \overline{\mathcal{A}_1 \circ \mathcal{PD}_4} : \pi_2^* \Psi_{\mathcal{PD}_6}(\tilde{f}, l_{\tilde{p}}) = 0\} = \Delta \mathcal{PD}_6. \tag{6.140}
\]
Observe that
\[
\overline{\mathcal{A}_1 \circ \mathcal{PD}_4} \cap \Delta(\mathcal{PD}_4 \cup \mathcal{PD}_5 \cup \mathcal{PE}_6) = \emptyset. \tag{6.141}
\]
This follows from (6.56), (6.74) and (6.108) combined with the fact that
\[(\mathcal{PD}_4 \cup \mathcal{PD}_5) \cap (\overline{\mathcal{PD}_5^\dagger} \cup \overline{\mathcal{PD}_6}) = \emptyset.\]
Eq. (6.141) implies that the lhs of (6.140) is a subset of its rhs. Next, we will simultaneously, prove the following two statements:
\[
\overline{\mathcal{A}_1 \circ \mathcal{PD}_4} \supset \Delta \mathcal{PD}_6, \tag{6.142}
\]
\[
\overline{\mathcal{A}_1 \circ \mathcal{PD}_5} \cap \Delta \mathcal{PD}_6 = \emptyset, \tag{6.143}
\]
Since $\overline{\mathcal{A}_1 \circ \mathcal{PD}_4}$ is a closed set, (6.142) implies that the rhs of (6.140) is a subset of its lhs.

Claim 6.24. Let $(\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \Delta \mathcal{PD}_6$. Then there exists a solution
\[
(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) \in (\mathcal{D} \times \mathbb{P}^2) \circ \mathcal{PD}_4
\]
near $(\tilde{f}, \tilde{p}, l_{\tilde{p}})$ to the set of equations
\[
\pi_1^* \psi_{A_0}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) = 0, \\
\pi_1^* \psi_{A_1}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) = 0, \quad \tilde{p}(t_1, t_2) \neq \tilde{p}(t_1). \tag{6.144}
\]
Moreover, such a solution lies in $\overline{\mathcal{A}_1 \circ \mathcal{PD}_4}$, i.e.,
\[
\pi_2^* \Psi_{\mathcal{PD}_6}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) \neq 0. \tag{6.145}
\]
In particular, $(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)})$ does not lie in $\overline{\mathcal{A}_1 \circ \mathcal{PD}_5}$.

Note that Claim 6.24 implies (6.142) and (6.143) simultaneously.

Proof. Choose homogeneous coordinates $[X : Y : Z]$ so that $\tilde{p} = [0 : 0 : 1]$ and let $\mathcal{U}_{\tilde{p}}$, $\pi_x$, $\pi_y$, $v_1$, $w$, $\eta$, $\eta_1$, $\eta_2$, $x_{t_1}$, $y_{t_1}$, $x_{t_2}$, $y_{t_2}$, $f_{ij}(t_1, t_2)$, $F$, $F_{x_{t_2}}$ and $F_{y_{t_2}}$ be exactly the same as defined in the proof of Claim 6.12, except for one difference: we take $(\tilde{f}(t_1, t_2), l_{\tilde{p}(t_1)})$ to be a point in $\overline{\mathcal{PD}_4}$. Since $(\tilde{f}(t_1, t_2), l_{\tilde{p}(t_1)}) \in \overline{\mathcal{PD}_4}$, we conclude that
\begin{equation}
\begin{aligned}
f_{00}(t_1, t_2) &= f_{10}(t_1, t_2) = f_{01}(t_1, t_2) = f_{11}(t_1, t_2) = f_{20}(t_1, t_2) = f_{02}(t_1, t_2) \\
&\quad = f_{30}(t_1, t_2) = 0.
\end{aligned}
\end{equation}

The functional equation (6.80) has a solution if and only if the following set of equations has a solution (as numbers):

\begin{equation}
F = 0, \quad F_{x_{t_2}} = 0, \quad F_{y_{t_2}} = 0, \quad (x_{t_2}, y_{t_2}) \neq (0, 0) \quad \text{(but small).} \quad (6.146)
\end{equation}

For the convenience of the reader, let us rewrite the expression for $F$:

\begin{equation}
F := \frac{f_{21}(t_1, t_2)}{2} x_{t_2}^2 y_{t_2} + \frac{f_{12}(t_1, t_2)}{2} x_{t_2} y_{t_2} + \frac{f_{03}(t_1, t_2)}{6} y_{t_2}^3 + \ldots.
\end{equation}

Since $(\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \Delta \mathcal{P} \mathcal{D}_6$, we conclude that

\begin{equation}
f_{21}, f_{40} = 0 \quad \text{and} \quad f_{12}, D^f_l \neq 0.
\end{equation}

A little bit of thought will reveal that the solutions to (6.146) are exactly the same as in (6.99), (6.100) and (6.101), with $f_{02}(t_1, t_2) = 0$. Since $\alpha \neq 0$, we conclude that $f_{21}(t_1, t_2) \neq 0$ for small but non-zero $\tilde{x}_{t_2}$. Hence (6.145) holds. □

**Corollary 6.25.** Let $\mathbb{W} \to \mathcal{D} \times \mathbb{P}^2 \times \mathbb{P} \mathbb{T}^2$ be a vector bundle such that the rank of $\mathbb{W}$ is same as the dimension of $\Delta \mathcal{P} \mathcal{D}_6$ and $Q : \mathcal{D} \times \mathbb{P}^2 \times \mathbb{P} \mathbb{T}^2 \to \mathbb{W}$ a generic smooth section. Suppose $(\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \Delta \mathcal{P} \mathcal{D}_6 \cap Q^{-1}(0)$. Then the section

\begin{equation}
\pi_2^* \psi_{\mathcal{P} \mathcal{D}_4} \oplus Q : \overline{\mathcal{A}_1 \circ \mathcal{P} \mathcal{D}_4} \longrightarrow \pi_2^*(\mathcal{L}_{\mathcal{P} \mathcal{D}_4}) \oplus \mathbb{W}
\end{equation}

vanishes around $(\tilde{f}, \tilde{p}, l_{\tilde{p}})$ with a multiplicity of 2.

**Proof.** Follows from the fact that the sections induced by $f_{21}$ and $f_{40}$ are transverse to the zero set over $\Delta \mathcal{P} \mathcal{D}_4$, the fact that $Q$ is generic and (6.100) (combined with $f_{02}(t_1, t_2) = 0$). Each branch of $\alpha := \sqrt{-\frac{\partial f_{02}(t_1, t_2)}{60 f_{12}(t_1, t_2)}}$ contributes with a multiplicity of 1. Hence, the total multiplicity is 2. □

This completes the proof of Lemma 6.3 (7). □

Before proceeding further, observe that (6.76) implies that

\begin{equation}
\Delta \mathcal{P} \mathcal{D}_6^{\vee} \subset \Delta \mathcal{P} \mathcal{D}_6^{\vee}.
\end{equation}

\textsuperscript{19} Take partial derivatives with respect to $f_{21}$ and $f_{40}$.
Proof of Lemma 6.3 (8). Follows from Lemma 6.3 (7), (6.147), Lemma 5.2, (5.1) and (5.2). □

Proof of Lemma 6.3 (9). It suffices to prove the following two statements:

\[
\begin{align*}
\{(\tilde{f}, \tilde{p}, l_p) &\in \overline{A_1 \circ PD_5} : \pi_2^*\Psi_{\mathcal{P}E_5}(\tilde{f}, \tilde{p}, l_p) \neq 0\} \\
&= \{(\tilde{f}, \tilde{p}, l_p) \in \Delta PD_7 : \pi_2^*\Psi_{\mathcal{P}E_5}(\tilde{f}, \tilde{p}, l_p) \neq 0\} \\
\{(\tilde{f}, \tilde{p}, l_p) &\in \overline{A_1 \circ PD_5} : \pi_2^*\Psi_{\mathcal{P}E_5}(\tilde{f}, \tilde{p}, l_p) = 0\} = \Delta PD_7.
\end{align*}
\]

Let us directly prove a more general version of (6.148):

Lemma 6.26. If \( k \geq 5 \) then

\[
\begin{align*}
\{(\tilde{f}, \tilde{p}, l_p) &\in \overline{A_1 \circ PD_k} : \pi_2^*\Psi_{\mathcal{P}E_5}(\tilde{f}, \tilde{p}, l_p) \neq 0\} \\
&= \{(\tilde{f}, \tilde{p}, l_p) \in \Delta PD_{k+2} : \pi_2^*\Psi_{\mathcal{P}E_5}(\tilde{f}, \tilde{p}, l_p) \neq 0\}.
\end{align*}
\]

Note that (6.148) is a special case of Lemma 6.26; take \( k = 5 \). We will prove the following two facts simultaneously:

\[
\begin{align*}
\{(\tilde{f}, \tilde{p}, l_p) &\in \overline{A_1 \circ PD_k} \supset \Delta PD_{k+2} \quad \forall k \geq 5, \quad (6.150) \\
\{(\tilde{f}, \tilde{p}, l_p) \in \overline{A_1 \circ PD_{k+1}} \cap \Delta PD_{k+2} = \emptyset \quad \forall k \geq 4. \quad (6.151)
\end{align*}
\]

It follows from Lemma 6.2 (6) that (6.150) and (6.151) imply Lemma 6.26. We will now prove the following claim:

Claim 6.27. Let \( (\tilde{f}, \tilde{p}, l_p) \in \Delta PD_{k+2} \) and \( k \geq 5 \). Then there exists a solution

\[
(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) \in (D \times \mathbb{P}^2) \circ PD_5
\]

near \( (\tilde{f}, \tilde{p}, l_p) \) to the set of equations

\[
\begin{align*}
\pi_{A_0}^* (\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) &= 0, & \pi_{A_1}^* (\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) &= 0, \\
\pi_{PD_5}^* (\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) &= 0, & \ldots, & \pi_{PD_5}^* (\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) &= 0, \\
\pi_{\mathcal{P}E_5}^* (\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) &\neq 0, & \tilde{p}(t_1, t_2) &\neq \tilde{p}(t_1). 
\end{align*}
\]

Moreover, any solution \( (\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) \) sufficiently close to \( (\tilde{f}, \tilde{p}, l_p) \) lies in \( \overline{A_1 \circ PD_k} \), i.e.,

\[
\pi_{PD_{k+1}}^* (\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) &\neq 0.
\]

In particular \( (\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) \) does not lie in \( \overline{A_1 \circ PD_{k+1}} \).
It is easy to see that Claim 6.27 implies (6.150) and (6.151) simultaneously for all $k \geq 5$. The fact that (6.151) holds for $k = 4$ is the content of (6.143).

**Proof.** Choose homogeneous coordinates $[X : Y : Z]$ so that $\bar{p} = [0 : 0 : 1]$ and let $\mathcal{U}_p$, $\pi_x, \pi_y, v_1, w, v, \eta, \eta_t, x_t, y_t, f_{ij}(t_1, t_2)$, $F, F_{x_t}$ and $F_{y_t}$ be exactly the same as defined in the proof of Claim 6.16. Hence

$$f_{10}(t_1, t_2) = f_{01}(t_1, t_2) = f_{11}(t_1, t_2) = f_{20}(t_1, t_2) = f_{02}(t_1, t_2) = f_{30}(t_1, t_2) = f_{21}(t_1, t_2) = 0.$$

Since $(\hat{f}, \bar{p}, l_p) \in \Delta PD_{k+2}$, we conclude that $f_{12}(t_1, t_2) \neq 0$. Hence, we can make a change of coordinates to write $F$ as

$$F = \hat{y}_{t_2}^2 \hat{x}_{t_2} + \frac{D_{6}^f(t_1, t_2)}{4!} \hat{x}_{t_2}^4 + \frac{D_{7}^f(t_1, t_2)}{5!} \hat{x}_{t_2}^5 + \cdots$$

The functional equation (6.152) has a solution if and only if the following set of equations has a solution (as numbers):

$$\hat{y}_{t_2}^2 \hat{x}_{t_2} + \frac{D_{6}^f(t_1, t_2)}{4!} \hat{x}_{t_2}^4 + \frac{D_{7}^f(t_1, t_2)}{5!} \hat{x}_{t_2}^5 + \cdots = 0, \quad \hat{y}_{t_2} \hat{x}_{t_2} = 0, \quad (\hat{y}_{t_2}, \hat{x}_{t_2}) \neq (0, 0) \quad \text{(but small)}.$$

It is easy to see that the solutions to (6.154) exist given by

$$D_{6}^f(t_1, t_2), \ldots, D_{k}^f(t_1, t_2) = 0,$$

$$D_{k+1}^f(t_1, t_2) = \frac{D_{k+3}^f(t_1, t_2)}{k(k+1)} \hat{x}_{t_2}^2 + O(\hat{x}_{t_2}^3),$$

$$D_{k+2}^f(t_1, t_2) = -\frac{2D_{k+3}^f(t_1, t_2)}{(k+1)} \hat{x}_{t_2} + O(\hat{x}_{t_2}^2),$$

$$\hat{y}_{t_2} = 0, \quad \hat{x}_{t_2} \neq 0 \quad \text{(but small)}.$$

By (6.155), it immediately follows that (6.153) holds. □

**Corollary 6.28.** Let $\mathbb{W} \to D \times \mathbb{P}^2 \times \mathbb{P} \mathbb{T} \mathbb{P}^2$ be a vector bundle such that the rank of $\mathbb{W}$ is same as the dimension of $\Delta PD_{k+2}$ and $Q : D \times \mathbb{P}^2 \times \mathbb{P} \mathbb{T} \mathbb{P}^2 \to \mathbb{W}$ a generic smooth section. Suppose $(\hat{f}, \bar{p}, l_p) \in \Delta PD_{k+2} \cap Q^{-1}(0)$. Then the section

$$\pi_2^{\ast} \Psi_{PD_{k+1}} \oplus Q : \Delta P D_{k} \to \pi_2^{\ast}(L_{PD_{k+1}}) \oplus \mathbb{W}$$

vanishes around $(\hat{f}, \bar{p}, l_p)$ with a multiplicity of 2.
Proof. This follows from the fact that the sections
\[
\pi_2^* \psi_{\mathcal{P}D_i} : \Delta \overline{\mathcal{P}D}_{i-1} - \pi_2^* \psi_{\mathcal{P}E_0}^{-1}(0) \to \pi_2^* \mathcal{P}D_i
\]
are transverse to the zero set for all \(6 \leq i \leq k + 2\), the fact that \(Q\) is generic and (6.155). \(\Box\)

Next, we will prove (6.149). The lhs of (6.149) is a subset of its rhs; this follows from (6.108) and the fact that \(\mathcal{P}D_5\) is a subset of \(\overline{\mathcal{P}D}_4\). To prove the converse, we will prove the following three statements simultaneously:
\[
\begin{align*}
\overline{A}_1 \circ \mathcal{P}D_5 &\supset \Delta \mathcal{P}E_7, \quad (6.156) \\
\overline{A}_1 \circ \mathcal{P}D_6 &\cap \Delta \mathcal{P}E_7 = \emptyset. \quad (6.157) \\
\overline{A}_1 \circ \mathcal{P}E_6 &\cap \Delta \mathcal{P}E_7 = \emptyset. \quad (6.158)
\end{align*}
\]
Since \(\overline{A}_1 \circ \mathcal{P}D_5\) is a closed set, (6.156) implies that the rhs of (6.149) is a subset of its lhs.

Claim 6.29. Let \((\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \Delta \mathcal{P}E_7\). Then there exist solutions
\[
(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) \in (D \times \mathbb{P}^2) \circ \mathcal{P}D_5
\]

near \((\tilde{f}, \tilde{p}, l_{\tilde{p}})\) to the set of equations
\[
\begin{align*}
\pi_1^* \psi_{\mathcal{A}_0}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) &= 0, \\
\pi_1^* \psi_{\mathcal{A}_1}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) &= 0, \quad \tilde{p}(t_1, t_2) \neq \tilde{p}(t_1). \quad (6.159)
\end{align*}
\]

Moreover, any such solution sufficiently close to \((\tilde{f}, \tilde{p}, l_{\tilde{p}})\) lies in \(\overline{A}_1 \circ \mathcal{P}D_5\), i.e.,
\[
\pi_2^* \psi_{\mathcal{P}D_6}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) \neq 0.
\]

In particular, \((\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)})\) does not lie in \(\overline{A}_1 \circ \mathcal{P}D_6\). Since
\[
\pi_2^* \psi_{\mathcal{P}E_6}(\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)}) \neq 0
\]
the solution \((\tilde{f}(t_1, t_2), \tilde{p}(t_1, t_2), l_{\tilde{p}(t_1)})\) does not lie in \(\overline{A}_1 \circ \mathcal{P}E_6\).

Note that Claim 6.29 implies (6.149) and (6.157) simultaneously. This completes the proof of Lemma 6.3 (9). \(\Box\)

Proof. Choose homogeneous coordinates \([X : Y : Z]\) so that \(\tilde{p} = [0 : 0 : 1]\) and let \(U_{\tilde{p}}, \pi_x, \pi_y, v_1, w, v, \eta, \eta_{t_1}, \eta_{t_2}, x_{t_1}, y_{t_1}, x_{t_2}, y_{t_2}, \hat{f}_{ij}(t_1, t_2), F, F_{x_{t_2}}\) and \(F_{y_{t_2}}\) be exactly the same
as defined in the proof of Claim 6.16 except for one difference: we take \((\tilde{f}(t_1, t_2), l_{\tilde{p}(t_1)})\) to be a point in \(\overline{P_D}_5\). Since \((\tilde{f}(t_1, t_2), l_{\tilde{p}(t_1)}) \in \overline{P_D}_5\), we conclude that

\[
f_{10}(t_1, t_2) = f_{01}(t_1, t_2) = f_{11}(t_1, t_2) = f_{20}(t_1, t_2) = f_{02}(t_1, t_2) = f_{30}(t_1, t_2) = f_{21}(t_1, t_2) = 0.
\]

Since \((\tilde{f}, \tilde{p}, l_{\tilde{p}}) \in \Delta \mathcal{E}_7\), we conclude that

\[
f_{03}, f_{31} \neq 0, \quad f_{12}, f_{40} = 0. \tag{6.160}
\]

The functional equation (6.159) has a solution if and only if the following set of equations has a solution (as numbers):

\[
F = 0, \quad F_{x_{t_2}} = 0, \quad F_{y_{t_2}} = 0, \quad (x_{t_2}, y_{t_2}) \neq (0, 0) \quad \text{(but small).} \tag{6.161}
\]

Let us now define

\[G := -8F + 2x_{t_2}F_{x_{t_2}} + 3y_{t_2}F_{y_{t_2}}.\]

We claim that \(x_{t_2} \neq 0\) and \(y_{t_2} \neq 0\); we will justify that at the end. Assuming that claim we conclude that solving (6.161) is equivalent to solving

\[
G = 0, \quad F_{x_{t_2}} = 0, \quad F_{y_{t_2}} = 0, \quad (x_{t_2}, y_{t_2}) \neq (0, 0) \quad \text{(but small).} \tag{6.162}
\]

Note that \(G\) is independent of \(f_{12}(t_1, t_2)\) and \(f_{40}(t_1, t_2)\). Hence, \(G\) is explicitly given by

\[G = \frac{P_{03}(x_{t_2}, y_{t_2})}{6} y_{t_2}^3 + \frac{P_{31}(x_{t_2}, y_{t_2})}{6} x_{t_2}^3 y + \kappa x_{t_2}^2 y_{t_2}^2 + P_{50}(x_{t_2}) x_{t_2}^5,
\]

where \(P_{03}(0, 0) = f_{03}(t_1, t_2)\) and \(P_{31}(0, 0) = f_{31}(t_1, t_2)\). Using the same argument as in [2] (where we give a necessary and sufficient criteria for a curve to have an \(\mathcal{E}_7\)-node), there exists a holomorphic function \(B(\hat{x}_{t_2})\) and constant \(\eta\) such that if we make the substitution

\[x_{t_2} = \hat{x}_{t_2} + \eta \hat{y}_{t_2}, \quad y_{t_2} = \hat{y}_{t_2} + B(\hat{x}_{t_2}) \hat{x}_{t_2}^2\]

then \(G\) is given by

\[G = \frac{\hat{P}_{03}(\hat{x}_{t_2}, \hat{y}_{t_2})}{6} \hat{y}_{t_2}^3 + \frac{\hat{P}_{31}(\hat{x}_{t_2}, \hat{y}_{t_2})}{6} \hat{x}_{t_2}^3 \hat{y}_{t_2},
\]

where \(\hat{P}_{03}(0, 0) = f_{03}(t_1, t_2)\) and \(\hat{P}_{31}(0, 0) = f_{31}(t_1, t_2)\). We claim that \(\hat{y}_{t_2} \neq 0\); we will justify that at the end. Assuming that claim, we conclude from \(G = 0\) that
\[ \dot{y}_{t_2} = u^3, \quad \dot{x}_{t_2} = \alpha u^2 + O(u^2) \]

where \( \alpha := \frac{3}{2} \frac{f_{03}(t_1, t_2)}{f_{31}(t_1, t_2)} \) a branch of the cuberoot.

Using this and the remaining two equations of (6.162) we conclude that

\[ f_{12}(t_1, t_2) = -\frac{f_{03}(t_1, t_2)}{3\alpha} u + O(u^2), \]
\[ f_{40}(t_1, t_2) = -\frac{4f_{31}(t_1, t_2)}{\alpha} u + O(u^2). \]  

(6.163)

It remains to show that \( x_{t_2} \neq 0, y_{t_2} \neq 0 \) and \( \dot{y}_{t_2} \neq 0 \). If \( x_{t_2} = 0 \), then \( F = 0 \) implies that \( f_{03}(t_1, t_2) = O(y_{t_2}) \), contradicting (6.160). Next, if \( y_{t_2} = 0 \) then \( F_{y_{t_2}} = 0 \) implies that \( f_{31}(t_1, t_2) = O(x_{t_2}) \), contradicting (6.160). Finally, if \( \dot{y}_{t_2} = 0 \), then \( F_{y_{t_2}} = 0 \) implies that

\[ f_{31}(t_1, t_2) = -6B(0)f_{12}(t_1, t_2) + O(x_{t_2}). \]

As \( f_{12}(t_1, t_2) \) and \( x_{t_2} \) go to zero, \( f_{31}(t_1, t_2) \) goes to zero, contradicting (6.160).

\[ \square \]

**Corollary 6.30.** Let \( \mathbb{W} \to \mathcal{D} \times \mathbb{P}^2 \times \mathbb{P} \mathcal{T} \mathbb{P}^2 \) be a vector bundle such that the rank of \( \mathbb{W} \) is same as the dimension of \( \Delta \mathcal{PE}_7 \) and \( Q : \mathcal{D} \times \mathbb{P}^2 \times \mathbb{P} \mathcal{T} \mathbb{P}^2 \to \mathbb{W} \) a generic smooth section. Suppose \((f, \tilde{\tilde{p}}, l_{\tilde{\tilde{p}}} ) \in \Delta \mathcal{PE}_7 \cap Q^{-1}(0) \). Then the sections

\[ \pi^*_2 \Psi_{PD_6} \oplus Q : \tilde{\tilde{A}}_1 \circ P D_5 \to \pi^*_2 (L_{PD_6}) \oplus \mathbb{W}, \]
\[ \pi^*_2 \Psi_{PE_6} \oplus Q : \tilde{\tilde{A}}_1 \circ P D_5 \to \pi^*_2 (L_{PE_6}) \oplus \mathbb{W} \]

vanish around \((f, \tilde{\tilde{p}}, l_{\tilde{\tilde{p}}} )\) with a multiplicity of 1.

**Proof.** Follows from the fact that the sections induced by \( f_{12} \) and \( f_{40} \) are transverse to the zero set over \( \Delta \mathcal{PD}_5 \),\(^{20} \) \( Q \) is generic and (6.163).

\[ \square \]

7. Euler class computation

We are ready to prove the recursive formulas stated in Section 3.

**Proof of (3.2).** Let \( Q : \mathcal{D} \times \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{W} \) be a generic smooth section to

\[ \mathbb{W} := \bigoplus_{i=1}^{\delta_d - (n+2)} \pi^*_D \gamma^*_D \bigoplus \bigoplus_{i=1}^{n} \pi^*_2 \gamma^*_2 \to \mathcal{D} \times \mathbb{P}^2 \times \mathbb{P}^2. \]

\(^{20} \)Take partial derivatives with respect to \( f_{12} \) and \( f_{40} \).
Note that
\[ N(A_1 A_1, n) = \langle e(W), [A_1 \circ A_1] \rangle = |\pm(A_1 \circ A_1) \cap Q^{-1}(0)|. \]

By Lemma 6.3 (1)
\[ \overline{A}_1 \times \mathbb{P}^2 = \overline{A}_1 \circ (D \times \mathbb{P}^2) = \overline{A}_1 \circ (D \times \mathbb{P}^2) \sqcup \Delta \overline{A}_1. \]
The sections
\[ \pi_2^* \psi_{A_0} : \overline{A}_1 \times \mathbb{P}^2 - \Delta \overline{A}_1 \longrightarrow \pi_2^* \mathcal{L}_{A_0}, \quad \pi_2^* \psi_{A_1} : \pi_2^* \psi_{A_0}^{-1}(0) \longrightarrow \pi_2^* \mathcal{V}_{A_1} \]
are transverse to the zero set. (cf. Proposition 5.3). Hence
\[ \langle e(\pi_2^* \mathcal{L}_{A_0}) e(\pi_2^* \mathcal{V}_{A_1}) e(W), [\overline{A}_1 \times \mathbb{P}^2] \rangle = N(A_1 A_1, n) + C_{\Delta \overline{A}_1}(\pi_2^* \psi_{A_0} \oplus \pi_2^* \psi_{A_1} \oplus \mathcal{Q}), \]
where \( C_{\Delta \overline{A}_1}(\pi_2^* \psi_{A_0} \oplus \pi_2^* \psi_{A_1} \oplus \mathcal{Q}) \) is the contribution of the section \( \pi_2^* \psi_{A_0} \oplus \pi_2^* \psi_{A_1} \oplus \mathcal{Q} \) to the Euler class from \( \Delta \overline{A}_1 \). The lhs of (7.2), as computed by splitting principle and a case by case check, is
\[ \langle e(\pi_2^* \mathcal{L}_{A_0}) e(\pi_2^* \mathcal{V}_{A_1}) e(W), [\overline{A}_1 \times \mathbb{P}^2] \rangle = N(A_1, 0) \times N(A_1, n). \]
In fact, the usual answer one arrives at is
\[ N(A_1, n) + 3(d - 1)N(A_1, n + 1) + 3(d - 1)^2 N(A_1, n + 2). \]

One then uses a result from [1]:
\[ N(A_1, n) = \begin{cases} 3(d - 1)^2, & \text{if } n = 0; \\ 3(d - 1), & \text{if } n = 1; \\ 1, & \text{if } n = 2; \\ 0, & \text{otherwise.} \end{cases} \]

Next, we compute \( C_{\Delta \overline{A}_1}(\pi_2^* \psi_{A_0} \oplus \pi_2^* \psi_{A_1} \oplus \mathcal{Q}) \). Note that \( \overline{A}_1 = A_1 \sqcup A_2 \). By Claim 6.6 we get that
\[ C_{\Delta \overline{A}_1}(\pi_2^* \psi_{A_0} \oplus \pi_2^* \psi_{A_1} \oplus \mathcal{Q}) = \langle e(\pi_2^* \mathcal{L}_{A_0}) e(W), [\Delta \overline{A}_1] \rangle = N(A_1, n) + dN(A_1, n + 1), \]
\[ C_{\Delta \overline{A}_2}(\pi_2^* \psi_{A_0} \oplus \pi_2^* \psi_{A_1} \oplus \mathcal{Q}) = 3N(A_2, n). \]

It is easy to see that (7.3), (7.5) and (7.6) prove (3.2). □
**Remark 7.1.** In [7] a different method is used to compute \( C_{\Delta A_1}(\pi_2^*\psi_{A_0} \oplus \pi_2^*\psi_{A_1} \oplus Q) \). The author of that paper later on pointed out this simpler method to the second author of this paper.

**Proof of (3.3) and (3.4).** Let \( \mathbb{W}^1_{n,m,2} \) and \( Q \) be as in (2.7) with \( k = 2 \). By definition, \( \mathcal{N}(A_1 \cap \mathcal{P} A_2, n, m) \) is the signed cardinality of the intersection of \( A_1 \cap \mathcal{P} A_2 \) with \( Q^{-1}(0) \). By Lemma 6.3 (2) we gather that

\[
\overline{A_1} \circ \overline{A_1'} = \overline{A_1} \circ \overline{A_1'} \cup \overline{A_1} \circ (\overline{A_1'} - \overline{A_1'}) \cup \Delta \overline{A_3} = \overline{A_1} \circ \overline{A_1'} \cup \overline{A_1} \circ \mathcal{P} A_2 \cup \Delta \overline{A_3} \quad \text{(by Lemma 6.1 (2)).}
\]

By Proposition 5.4, the section

\[
\pi_2^*\psi_{\mathcal{P} A_2} : \overline{A_1} \circ \overline{A_1'} \to \pi_2^*\mathcal{V}_{\mathcal{P} A_2}
\]

vanishes on \( A_1 \cap \mathcal{P} A_2 \) transversely. Hence, the zeros of the section

\[
\pi_2^*\psi_{\mathcal{P} A_2} \oplus Q : \overline{A_1} \circ \overline{A_1'} \to \pi_2^*\mathcal{V}_{\mathcal{P} A_2} \oplus \mathbb{W}^1_{n,m,2},
\]

restricted to \( A_1 \cap \mathcal{P} A_2 \) counted with a sign, is our desired number. In other words

\[
\langle e(\pi_2^*\mathcal{V}_{\mathcal{P} A_2})e(\mathbb{W}^1_{n,m,2}), [\overline{A_1} \circ \overline{A_1']}] \rangle = \mathcal{N}(A_1 \cap \mathcal{P} A_2, n, m) + C_{\Delta \overline{A_3}}(\pi_2^*\psi_{\mathcal{P} A_2} \oplus Q)
\]

where \( C_{\Delta \overline{A_3}}(\pi_2^*\psi_{\mathcal{P} A_2} \oplus Q) \) is the contribution of the section to the Euler class from \( \Delta \overline{A_3} \).

Note that \( \pi_2^*\psi_{\mathcal{P} A_2} \oplus Q \) vanishes only on \( \mathcal{P} A_3 \) and \( \hat{D}_4 \) and not on the entire \( \overline{A_3} \). By Corollaries 6.9 and 6.10, the contribution from \( \mathcal{P} A_3 \) and \( \hat{D}_4 \) are 2 and 3 respectively. This proves the claim. \( \square \)

**Remark 7.2.** In the above proof we are using Lemma B.2 with \( M := \overline{A_1} \circ \overline{A_1'} \). However, in this case \( M \) is not a smooth manifold; it is only a pseudocycle. Lemma B.2 is actually true even when \( M \) happens to be a pseudocycle.

**Remark 7.3.** A completely different method is used in [7] to compute \( \mathcal{N}(A_1 \cap A_2, n) \); instead of removing the cusp, the node is removed. In fact, all the numbers \( \mathcal{N}(A_1 \cap \xi_k, n) \) can also be computed by removing the node, instead of removing the \( \xi_k \) singularity. However, in order to obtain a recursive formula for the number of degree-\( d \) curves through \( \delta_d - (\delta + k) \) generic points and having \( \delta \)-nodes and one singularity of type \( \xi_k \), we have to apply the method employed in this paper (i.e., we have to remove the \( \xi_k \)-singularity, not the node). This observation is again due to Aleksey Zinger.

**Proof of (3.5).** Let \( \mathbb{W}^1_{n,m,3} \) and \( Q \) be as in (2.7) with \( k = 3 \). By Lemma 6.3 (3) we have
\[ \overline{A_1 \circ P A_2} = A_1 \circ P A_2 \sqcup A_1 \circ (P A_2 - P A_2) \sqcup (\Delta P A_1 \cup \Delta \overline{D}_5^#), \]
\[ = A_1 \circ P A_2 \sqcup A_1 \circ (P A_3 - P A_3) \sqcup (\Delta P A_4 \cup \Delta \overline{D}_5^#) \]

where the last equality follows from Lemma 6.1 (8). By Proposition 5.6, the section
\[ \pi^*_2 \Psi_{P A_3} \oplus Q : \overline{A_1 \circ P A_2} \longrightarrow \pi^*_2 L_{P A_3} \oplus W^1_{n,m,3} \]
vanishes transversely on \( A_1 \circ P A_3 \). By definition, it does not vanish on \( A_1 \circ \overline{D}_4^# \). By Corollary 6.13, the contribution to the Euler class from the points of \( \Delta P A_4 \) is 2. Furthermore, by definition the section does not vanish on \( \Delta \overline{D}_5^# \). Hence
\[ \langle e(\pi^*_2 L_{P A_3} \oplus W^1_{n,m,3}), [\overline{A_1 \circ P A_2}] \rangle = \mathcal{N}(A_1 P A_3, n, m) + 2N(P A_4, n, m) \]
which proves the equation. \( \square \)

**Proof of (3.6).** Let \( W^1_{n,m,4} \) and \( Q \) be as in (2.7) with \( k = 4 \). By Lemma 6.3 (3) we have
\[ \overline{A_1 \circ P A_3} = A_1 \circ P A_3 \sqcup A_1 \circ (P A_3 - P A_3) \sqcup (\Delta P A_5 \cup \Delta \overline{D}_5^Y), \]
\[ = A_1 \circ P A_3 \sqcup A_1 \circ (P A_4 - P A_4) \sqcup (\Delta P A_5 \cup \Delta \overline{D}_5^Y) \]
where the last equality follows from Lemma 6.1 (9). By Proposition 5.6, the section
\[ \pi^*_2 \Psi_{P A_4} \oplus Q : \overline{A_1 \circ P A_3} \longrightarrow \pi^*_2 L_{P A_4} \oplus W^1_{n,m,4} \]
vanishes transversely on \( A_1 \circ P A_4 \). It is easy to see that it does not vanish on \( A_1 \circ \overline{P D}_4 \). By Corollary 6.13, the contribution to the Euler class from the points of \( \Delta P A_5 \) is 2. Moreover, the section does not vanish on \( \Delta \overline{P D}_5^Y \). Hence
\[ \langle e(\pi^*_2 L_{P A_4} \oplus W^1_{n,m,4}), [\overline{A_1 \circ P A_3}] \rangle = \mathcal{N}(A_1 P A_4, n, m) + 2N(P A_5, n, m) \]
which proves the equation. \( \square \)

**Proof of (3.7).** Let \( W^1_{n,m,5} \) and \( Q \) be as in (2.7) with \( k = 5 \). By Lemma 6.3 (3) we have
\[ \overline{A_1 \circ P A_4} = A_1 \circ P A_4 \sqcup A_1 \circ (P A_4 - P A_4) \sqcup (\Delta P A_5 \cup \Delta \overline{D}_5^Y \cup \Delta \overline{P E}_6), \]
\[ = A_1 \circ P A_4 \sqcup A_1 \circ (P A_5 - P A_5) \sqcup (\Delta P A_6 \cup \Delta \overline{D}_5^Y \cup \Delta \overline{P E}_6) \]
where the last equality follows from Lemma 6.1 (10). By Proposition 5.6, the section
\[ \pi^*_2 \Psi_{P A_5} \oplus Q : \overline{A_1 \circ P A_4} \longrightarrow \pi^*_2 L_{P A_5} \oplus W^1_{n,m,5} \]
vanishes transversely on $A_1 \circ \mathcal{P}A_5$. In [1] we show that this section vanishes on $A_1 \circ \mathcal{PD}_5$ with a multiplicity of 2. By Corollaries 6.13 and 6.20, the contribution to the Euler class from the points of $\Delta \mathcal{P}A_6$ and $\Delta \mathcal{PE}_6$ are 2 and 5 respectively. Since the dimension of $\mathcal{PD}_7$ is one less than the rank of $\pi_2^sL_{\mathcal{P}A_5} \oplus W_{n,m,5}$ and $Q$ is generic, the section does not vanish on $\Delta \mathcal{PD}_7$. Since $\overline{\mathcal{PD}}_7$ is a subset of $\Delta \mathcal{PD}_7$ (by (6.103)), the section does not vanish on $\overline{\mathcal{PD}}_7$ either. Hence

$$
\langle e(\pi_2^sL_{\mathcal{P}A_5} \oplus W^1_{n,m,5}), [\overline{A_1 \circ \mathcal{P}A_4}] \rangle = N(A_1 \mathcal{P}A_5, n, m) + 2N(A_1 \mathcal{PD}_5, n, m) \\
+ 2N(\mathcal{P}A_6, n, m) + 5N(\mathcal{PE}_6, n, m)
$$

which proves the equation. □

**Proof of (3.8).** Let $W_{n,m,6}^1$ and $Q$ be as in (2.7) with $k = 6$. By Lemma 6.3 (6) we have

$$
\overline{\mathcal{A}_1 \circ \mathcal{P}A_5} = \overline{A_1 \circ \mathcal{P}A_5} \cup \overline{A_1 \circ (\mathcal{P}A_5 - \mathcal{P}A_5)} \cup (\Delta \mathcal{P}A_7 \cup \Delta \mathcal{PD}_8 \cup \Delta \mathcal{PE}_7) \\
= \overline{A_1 \circ \mathcal{P}A_5} \cup \overline{A_1 \circ (\mathcal{P}A_6 \cup \mathcal{PD}_6 \cup \mathcal{PE}_6)} \cup (\Delta \mathcal{P}A_7 \cup \Delta \mathcal{PD}_8 \cup \Delta \mathcal{PE}_7)
$$

where the last equality follows from Lemma 6.1 (11). By Proposition 5.6, the section

$$
\pi_2^s\Psi_{\mathcal{P}A_6} \oplus Q : \overline{\mathcal{A}_1 \circ \mathcal{P}A_5} \rightarrow \pi_2^sL_{\mathcal{P}A_6} \oplus W_{n,m,6}^1
$$

vanishes transversely on $A_1 \circ \mathcal{P}A_6$. In [1] we show that this section vanishes on $A_1 \circ \mathcal{PD}_6$ with a multiplicity of 4. By Corollaries 6.13 and 6.23, the contribution to the Euler class from the points of $\Delta \mathcal{P}A_7$ and $\Delta \mathcal{PE}_7$ are 2 and 6 respectively. Since the dimension of $\mathcal{PD}_8$ is one less than the rank of $\pi_2^sL_{\mathcal{P}A_6} \oplus W_{n,m,6}^1$ and $Q$ is generic, the section does not vanish on $\Delta \mathcal{PD}_8$. Since $\Delta \mathcal{PD}_8^\ast$ is a subset of $\Delta \mathcal{PD}_8$ (by (6.118)), the section does not vanish on $\Delta \mathcal{PD}_8^\ast$ either. Hence

$$
\langle e(\pi_2^sL_{\mathcal{P}A_6} \oplus W^1_{n,m,6}), [\overline{A_1 \circ \mathcal{P}A_5}] \rangle = N(A_1 \mathcal{P}A_6, n, m) + 4N(A_1 \mathcal{PD}_6, n, m) \\
+ 2N(\mathcal{P}A_7, n, m) + 6N(\mathcal{PE}_7, n, m)
$$

which proves the equation. □

**Proof of (3.9).** Let $W_{n,0,4}^1$ and $Q$ be as in (2.7) with $k = 4$ and $m = 0$. By Lemma 6.3 (3) we have

$$
\overline{A_1 \circ \mathcal{P}A_3} = \overline{A_1 \circ \mathcal{P}A_3} \cup \overline{A_1 \circ (\mathcal{P}A_3 - \mathcal{P}A_3)} \cup (\Delta \mathcal{P}A_5 \cup \Delta \mathcal{PD}_5^\ast) \\
= \overline{A_1 \circ \mathcal{P}A_3} \cup \overline{A_1 \circ (\mathcal{P}A_4 \cup \mathcal{PD}_4)} \cup (\Delta \mathcal{P}A_5 \cup \Delta \mathcal{PD}_5^\ast)
$$

where the last equality follows from Lemma 6.1 (9). By Proposition 5.6, the section

$$
\pi_2^s\Psi_{\mathcal{PD}_4} \oplus Q : \overline{A_1 \circ \mathcal{P}A_3} \rightarrow \pi_2^sL_{\mathcal{PD}_4} \oplus W_{n,m,4}^1
$$
vanishes transversely on $\mathcal{A}_1 \circ \mathcal{P} \mathcal{D}_4$. It is easy to see that the section does not vanish on $\mathcal{A}_1 \circ \mathcal{P} \mathcal{A}_4$. By Corollaries 6.13 and 6.17, the contribution to the Euler class from the points of $\Delta \mathcal{P} \mathcal{A}_5$ and $\mathcal{P} \mathcal{D}_5^\vee$ are 2 and 2 respectively. Hence

$$
\langle e(\pi_2^* L_{\mathcal{P} \mathcal{D}_4} \oplus \mathbb{W}_{n,0,4}^1), [\mathcal{A}_1 \circ \mathcal{P} \mathcal{A}_3] \rangle = N(\mathcal{A}_1 \mathcal{P} \mathcal{A}_4, n, 0) + 2N(\mathcal{P} \mathcal{A}_5, n, 0)
$$

$$
+ 2\langle e(\mathbb{W}_{n,0,4}^1), [\mathcal{P} \mathcal{D}_5^\vee] \rangle.
$$

(7.7)

Since the map $\pi : \mathcal{P} \mathcal{D}_5^\vee \to \mathcal{D}_5$ is one to one, we conclude that

$$
\langle e(\mathbb{W}_{n,0,4}^1), [\mathcal{P} \mathcal{D}_5^\vee] \rangle = N(\mathcal{D}_5, n).
$$

(7.8)

Eqs. (7.8) and (7.7) imply (3.9). $\square$

**Proof of (3.10).** Let $\mathbb{W}_{n,1,4}^1$ and $\mathcal{Q}$ be as in (2.7) with $k = 4$ and $m = 1$. By Lemma 6.3 (8) we have

$$
\mathcal{A}_1 \circ \hat{\mathcal{D}}^\# = \mathcal{A}_1 \circ \hat{\mathcal{D}}^\# \sqcup \mathcal{A}_1 \circ (\hat{\mathcal{D}}^\# - \hat{\mathcal{D}}_4) \sqcup (\Delta \hat{\mathcal{D}}^\#_6)
$$

implies

$$
\mathcal{A}_1 \circ \hat{\mathcal{D}}^\#_4 = \mathcal{A}_1 \circ \hat{\mathcal{D}}^\#_4 \sqcup \mathcal{A}_1 \circ (\hat{\mathcal{D}}^\#_4 - \hat{\mathcal{D}}^\#_4) \sqcup (\Delta \hat{\mathcal{D}}^\#_6)
$$

(since $\hat{\mathcal{D}}_4 = \hat{\mathcal{D}}^\#_4$ and $\hat{\mathcal{D}}_6 = \hat{\mathcal{D}}^\#_6$)

$$
= \mathcal{A}_1 \circ \hat{\mathcal{D}}^\#_4 \sqcup \mathcal{A}_1 \circ \mathcal{P} \mathcal{D}_4 \sqcup (\Delta \hat{\mathcal{D}}^\#_6) \text{ (by definition)}.
$$

By Proposition 5.9, the section

$$
\pi_2^* \psi_{\mathcal{P} \mathcal{A}_3} \oplus \mathcal{Q} : \mathcal{A}_1 \circ \hat{\mathcal{D}}^\#_4 \longrightarrow \pi_2^* L_{\mathcal{P} \mathcal{A}_3} \oplus \mathbb{W}_{n,m,4}^1
$$

vanishes transversely on $\mathcal{A}_1 \circ \mathcal{P} \mathcal{D}_4$. By definition, the section does not vanish on $\hat{\mathcal{D}}^\#_6$. Hence

$$
\langle e(\pi_2^* L_{\mathcal{P} \mathcal{A}_3} \oplus \mathbb{W}_{n,1,4}^1), [\mathcal{A}_1 \circ \hat{\mathcal{D}}^\#_4] \rangle = N(\mathcal{A}_1 \mathcal{P} \mathcal{D}_4, n, 1)
$$

which proves the equation. $\square$

**Proof of (3.11).** Let $\mathbb{W}_{n,m,5}^1$ and $\mathcal{Q}$ be as in (2.7) with $k = 5$. By Lemma 6.3 (7) we have

$$
\mathcal{A}_1 \circ \mathcal{P} \mathcal{D}_4 = \mathcal{A}_1 \circ \mathcal{P} \mathcal{D}_4 \sqcup \mathcal{A}_1 \circ (\mathcal{P} \mathcal{D}_4 - \mathcal{P} \mathcal{D}_4) \sqcup (\Delta \mathcal{P} \mathcal{D}_6 \cup \Delta \mathcal{P} \mathcal{D}_6^\vee)
$$

$$
= \mathcal{A}_1 \circ \mathcal{P} \mathcal{D}_4 \sqcup \mathcal{A}_1 \circ (\mathcal{P} \mathcal{D}_5 \cup \mathcal{P} \mathcal{D}_5^\vee) \sqcup (\Delta \mathcal{P} \mathcal{D}_6 \cup \Delta \mathcal{P} \mathcal{D}_6^\vee)
$$

where the last equality follows from Lemma 6.1 (4). By Proposition 5.5, the section

$$
\pi_2^* \psi_{\mathcal{P} \mathcal{D}_5} \oplus \mathcal{Q} : \mathcal{A}_1 \circ \mathcal{P} \mathcal{D}_4 \longrightarrow \pi_2^* L_{\mathcal{P} \mathcal{D}_5} \oplus \mathbb{W}_{n,m,5}^1
$$
vanishes transversely on \( A_1 \circ P \mathcal{D}_5 \). Moreover, it does not vanish on \( \overline{A}_1 \circ P \mathcal{D}_5^\vee \) by definition. By Corollary 6.25, the contribution to the Euler class from the points of \( \Delta \mathcal{P} \mathcal{D}_6 \) is 2. The section does not vanish on \( \Delta \mathcal{P} \mathcal{D}_6^{\vee s} \) by definition. Since the dimension of \( \mathcal{P} \mathcal{D}_6^\vee \) is same as the dimension of \( \mathcal{P} \mathcal{D}_6 \) and \( Q \) is generic, by (6.147), the section does not vanish on \( \Delta \mathcal{P} \mathcal{D}_6^{\vee s} \). Hence

\[
\langle e(\pi_2^* L_{P \mathcal{D}_5} \oplus \mathbb{W}_{n,m,6}^1), \overline{[A_1 \circ P \mathcal{D}_5]} \rangle = N(A_1 \mathcal{P} \mathcal{D}_5, n, m) + 2N(\mathcal{P} \mathcal{D}_6, n, m)
\]

which proves the equation. \( \square \)

**Proof of (3.13) and (3.14).** Let \( \mathbb{W}_{n,m,6}^1 \) and \( Q \) be as in (2.7) with \( k = 6 \). By Lemma 6.3 (9) we have

\[
\overline{A}_1 \circ P \mathcal{D}_5 = \overline{A}_1 \circ P \mathcal{D}_5 \sqcup \overline{A}_1 \circ (P \mathcal{D}_5 - P \mathcal{D}_5) \sqcup (\Delta \mathcal{P} \mathcal{D}_7 \cup \Delta \mathcal{P} \mathcal{E}_7)
\]

\[
= \overline{A}_1 \circ P \mathcal{D}_5 \sqcup \overline{A}_1 \circ (P \mathcal{D}_6 \cup P \mathcal{E}_6) \sqcup (\Delta \mathcal{P} \mathcal{D}_7 \cup \Delta \mathcal{P} \mathcal{E}_7)
\]

where the last equality follows from Lemma 6.1 (6). By Propositions 5.7 and 5.8, the sections

\[
\pi_2^* \Psi_{P \mathcal{D}_6} \oplus Q : \overline{A}_1 \circ P \mathcal{D}_5 \longrightarrow \pi_2^* L_{P \mathcal{D}_6} \oplus \mathbb{W}_{n,m,6}^1,
\]

\[
\pi_2^* \Psi_{P \mathcal{E}_6} \oplus Q : \overline{A}_1 \circ P \mathcal{D}_5 \longrightarrow \pi_2^* L_{P \mathcal{E}_6} \oplus \mathbb{W}_{n,m,6}^1
\]

vanishes transversely on \( A_1 \circ P \mathcal{D}_6 \) and \( A_1 \circ P \mathcal{E}_6 \) respectively. Moreover, they do not vanish on \( A_1 \circ P \mathcal{E}_6 \) and \( A_1 \circ P \mathcal{D}_6 \) respectively. By Corollaries 6.28 and 6.30 the contribution of the section \( \pi_2^* \Psi_{P \mathcal{D}_6} \oplus Q \) to the Euler class from the points of \( \Delta \mathcal{P} \mathcal{D}_7 \) and \( \Delta \mathcal{P} \mathcal{E}_7 \) are 2 and 1 respectively. By Corollary 6.30, the contribution of the section \( \pi_2^* \Psi_{P \mathcal{E}_6} \oplus Q \) from the points of \( \Delta \mathcal{P} \mathcal{E}_7 \) is 1; moreover it does not vanish on \( \Delta \mathcal{P} \mathcal{D}_7 \). Hence

\[
\langle e(\pi_2^* L_{P \mathcal{D}_6} \oplus \mathbb{W}_{n,m,6}^1), \overline{[A_1 \circ P \mathcal{D}_5]} \rangle = N(A_1 \mathcal{P} \mathcal{D}_6, n, m) + 2N(\mathcal{P} \mathcal{D}_7, n, m) + N(\mathcal{P} \mathcal{E}_7, n, m)
\]

\[
\langle e(\pi_2^* L_{P \mathcal{E}_6} \oplus \mathbb{W}_{n,m,6}^1), \overline{[A_1 \circ P \mathcal{D}_5]} \rangle = N(A_1 \mathcal{P} \mathcal{E}_6, n, m) + N(\mathcal{P} \mathcal{E}_7, n, m)
\]

which prove Eqs. (3.12) and (3.13). \( \square \)

**Conflict of interest statement**

The authors have no conflict of interest to declare.

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One of the crucial results of this paper is to compute the closure of relevant spaces, i.e., to stratify the space \( \overline{A}_1 \circ P \mathcal{X}_k \) as described in (2.6). A key step there is to observe
that certain sections are transverse to the zero set and utilize them to describe the neighborhood of a point. The second author is indebted to Aleksey Zinger for sharing his understanding of transversality and explaining this crucial idea to him (i.e., how to describe the neighborhood of a point using transversality of bundle sections). In addition, the second author is also grateful to Aleksey Zinger for suggesting several non-trivial low degree checks to verify our formulas. One of those low degree checks proved to be crucial in figuring out a mistake the second author had made earlier. The authors are grateful to Dennis Sullivan for sharing his perspective on this problem and indicating its connection to other areas of mathematics. Finally, the second author is grateful to IMSc for its hospitality, where this paper was written.

Appendix A. Low degree checks

Verification of the number \( N(A_1A_1, 0) = 3(d - 2)(d - 1)(3d^2 - 3d - 11): \)
d = 2: The only way a conic can have 2 nodes is if it is a double line. There are no double lines through 3 generic points.
d = 3: The only way a cubic can have 2 nodes is if it breaks into a line and a conic. Hence the number of cubics with 2 unordered nodes is \( \binom{3}{2} = 3 \).
Verification of the number \( N(A_1A_2, 0) = 12(d - 3)(3d^3 - 6d^2 - 11d + 18): \)
d = 3: There are no cubics with one node and one cusp.
Verification of the number \( N(A_1A_3, 0) = 6(d - 3)(25d^3 - 71d^2 - 122d + 280): \)
d = 3: There are no cubics with one node and one tacnode.
d = 4: There are two possibilities here. The curve could break into a line and a cubic. The number of lines through a given point and tangent to a fixed cubic is 6. The number of cubics through 8 points, tangent to a given line is 4. Hence the total number of quartics with one node and one tacnode that breaks into a line and a cubic is

\[
\binom{10}{1} \times 6 + \binom{10}{2} \times 4 = 240.
\]

The number of genus zero quartics with one node and one tacnode is 1296 (cf. [8, p. 91]). Hence the total number of quartics with one node and one tacnode is 1536.
Verification of the number \( N(A_1D_1, n) \) and \( N(A_1PD_4, n, 1): \)
d = 4: These numbers can be verified by direct geometric means for all values of \( n \) in the case of quartics (cf. [2]).

Appendix B. Standard facts about Chern classes and projectivized bundle

Lemma B.1. (See [6, Theorem 14.10].) Let \( \mathbb{P}^n \) be the \( n \)-dimensional complex projective space and \( \gamma \rightarrow \mathbb{P}^n \) the tautological line bundle. Then the total Chern class \( c(T\mathbb{P}^n) \) is given by \( c(T\mathbb{P}^n) = (1 + c_1(\gamma^*))^{n+1} \).
Lemma B.2. (See [3, p. 270].) Let $V \to M$ be a complex vector bundle, of rank $k$, over a smooth manifold $M$ and $\pi : \mathbb{P}V \to M$ the projectivization of $V$. Let $\tilde{\gamma} \to \mathbb{P}V$ be the tautological line bundle over $\mathbb{P}V$ and $\lambda = c_1(\tilde{\gamma}^*)$. There is a linear isomorphism

$$H^*(\mathbb{P}V; \mathbb{Z}) \cong H^*(M; \mathbb{Z}) \otimes H^*(\mathbb{P}^{k-1}; \mathbb{Z}) \quad (B.1)$$

and an isomorphism of rings

$$H^*(\mathbb{P}V; \mathbb{Z}) \cong \frac{H^*(M; \mathbb{Z})[\lambda]}{\langle \lambda^k + \lambda^{k-1}\pi^*c_1(V) + \lambda^{k-2}\pi^*c_2(V) + \cdots + \pi^*c_k(V) \rangle}. \quad (B.2)$$

In particular, if $\omega \in H^*(M; \mathbb{Z})$ is a top cohomology class then

$$\langle \pi^*(\omega)\lambda^{k-1}, [\mathbb{P}V] \rangle = \langle \omega, [M] \rangle,$$

i.e., $\lambda^{k-1}$ is a cohomology extension of the fibre.

References