Enumeration of curves with singularities: further details

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**Introduction**

In this document, we provide some further details that were omitted in our papers [1] and [2]. These include general position arguments, showing why certain sections are well defined, why certain sections are transverse to the zero set and low degree checks.
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Chapter 1

Details omitted in [1]

1.1 General position argument

We denote the space of degree $d$-curves in $\mathbb{P}^2$ by $\mathcal{D}$. It follows that $\mathcal{D} \cong \mathbb{P}^{\delta_d}$, where $\delta_d = d(d + 3)/2$. Let $\gamma_{\mathbb{P}^2} \rightarrow \mathbb{P}^2$ be the tautological line bundle. A homogeneous degree $d$-polynomial $f$ (in 3 variables) induces a holomorphic section of the line bundle $\gamma_{\mathbb{P}^2}^d \rightarrow \mathbb{P}^2$. If $f$ is non-zero, then we will denote its equivalence class in $\mathcal{D}$ by $\tilde{f}$. Similarly, if $p$ is a non-zero vector in $\mathbb{C}^3$, we will denote its equivalence class in $\mathbb{P}^2$ by $\tilde{p}$.

Given a singularity of type $X_k$, we also denote the space of degree $d$-curves with a marked point $\tilde{p}$ such that the curve has a singularity of type $X_k$ at $\tilde{p}$, by the symbol $X_k$, i.e.,

$$X_k := \{(\tilde{f}, \tilde{p}) \in \mathcal{D} \times \mathbb{P}^2 : f \text{ has a singularity of type } X_k \text{ at the point } \tilde{p}\}.$$ 

Finally, let $\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_{\delta_d - (k + n)}$ be $\delta_d - (k + n)$ generic points in $\mathbb{P}^2$ and $L_1, L_2, \ldots, L_n$ be $n$ generic lines in $\mathbb{P}^2$. Define the following sets

$$H_i := \{ \tilde{f} \in \mathcal{D} : f(p_i) = 0 \}, \quad H_i^* := \{ \tilde{f} \in \mathcal{D} : f(p_i) = 0, \nabla f|_{p_i} \neq 0 \}$$

$$\hat{H}_i := H_i \times \mathbb{P}^2, \quad \hat{H}_i^* := H_i^* \times \mathbb{P}^2 \quad \text{and} \quad \hat{L}_i := \mathcal{D} \times L_i. \quad (1.1.1)$$

We will now give a proof of Lemma 2.6 in [1]; we restate it here with a precise bound on $d$.

Lemma 1.1.1. (cf. Lemma 2.6, [1]) Let $\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_{\delta_d - (k + n)}$ be $\delta_d - (k + n)$ generic points in $\mathbb{P}^2$ and $L_1, L_2, \ldots, L_n$ be $n$ generic lines in $\mathbb{P}^2$. Let $\hat{H}_i, \hat{H}_i^*$ and $\hat{L}_i$ be as defined in (1.1.1). Then

$$X_k \cap \hat{H}_1 \cap \ldots \cap \hat{H}_{\delta_d - (k + n)} \cap \hat{L}_1 \cap \ldots \cap \hat{L}_n = X_k \cap \hat{H}_1^* \cap \ldots \cap \hat{H}_{\delta_d - (k + n)}^* \cap \hat{L}_1 \cap \ldots \cap \hat{L}_n$$

and every intersection is transverse, provided $d \geq C_{X_k}$ and $k \geq 1$.

Remark 1.1.2. We recall that

$$C_{A_k} = k + 1, \quad C_{D_k} = k - 1, \quad C_{E_6} = 4, \quad C_{E_7} = 4.$$ 

Remark 1.1.3. In the subsequent discussion, when we make a statement involving a singularity of type $X_k$, it will assumed that $k \geq 1$.

\(^1\)In [1] we use the symbol $\tilde{A}$ to denote the equivalence class of $A$ instead of the standard $[A]$, which makes some of the calculations easier to read.
It suffices to prove this for the case $n = 0$. The proofs for $n = 1$ and $n = 2$ are identical. For $n > 2$ the Lemma is trivially true, since both the sets are empty.

First let us recall a standard fact from differential topology.

**Theorem 1.1.4. (Families Transversality Theorem)**

Let $\pi : \xi \to B$ be a smooth fiber bundle with fiber $X$. If $\varphi : \xi \to Y$ is a smooth map between smooth manifolds such that $\xi$ is transverse to a fixed submanifold $Z \subseteq Y$ then $\pi^{-1}(b)$ is transverse to $Z$ for almost all points $b$ in $B$.

**Proof:** Fix $z_0 \in Z, (b_0, x_0) \in \xi$ and $\varphi(b_0, x_0) = z_0$. Choose coordinate neighbourhoods $V$ of $z_0$ in $Y$, i.e.,

$$(\alpha, \beta) : V \to \mathbb{R}^s \times \mathbb{R}^m, z_0 \mapsto (0, 0)$$

and $V \cong \{(\alpha, \beta) | \|\alpha\| \leq 1, \|\beta\| \leq 1\}$ such that $V \cap Z \cong \{(\eta, 0) \text{ s.t. } \|\eta\| \leq 1\}$. Choose a product neighbourhood $U_0 \times X_0$ of $(b_0, x_0)$ such that $\varphi(U_0 \times X_0) \subseteq V$. On this neighbourhood $\varphi$ is transverse to $Z$ if and only if $0$ is a regular value of $\alpha \circ \varphi$.

Without loss of generality we can assume that $Z = \{z\}$ is a point. Since we’re making an almost all type of statement, if we work on a local chart in $Z$ then it reduces to the case of a point. We can cover $Z$ by a countable number of charts and the union of a countable number of measure zero sets is again measure zero.

Set $M \equiv \varphi^{-1}(z) \subset \xi$ to be a smooth submanifold.

Note that $b$ is not a value of $\pi|_M$ if and only if $z$ is not a value of

$$\varphi_b := \varphi|_{\pi^{-1}(b)} : \pi^{-1}(b) \to Y.$$  

Moreover, $b$ is a regular value of $\pi|_M$ (and actually a value) if and only if

$$(\pi|_M)_* : T_{(b,x)}M \longrightarrow T_bB$$

is surjective for any $(b,x) \in M$.

We have the following grid:

- $0 \longrightarrow T_x X \longrightarrow \xi \longrightarrow Y \longrightarrow 0$
- $0 \longrightarrow T_{(b,x)} M \longrightarrow T_{(b,x)} \xi \longrightarrow T_z Y \longrightarrow 0$
- $T_b B$
In any kind of grid like this, where the associated objects are vector spaces, we can infer

\[(\varphi_b)_* \text{ is surjective at } x \in X \iff (\pi|_M)_* \text{ is surjective at } (b, x) \in M.\]

Notice that \(b\) is a regular value of \(\pi|_M\) precisely when \((\varphi_b)_*\) is surjective for any \(x \in X\). This happens exactly when \(z\) is a regular value of \(\varphi_b : \pi^{-1}(b) \to Y\). By Sard’s theorem, almost every \(b \in B\) is a regular value of \(\pi|_M\). \(\square\)

**Corollary 1.1.5.** Let \(\pi : \xi \to B\) be a bundle and \(\varphi : \xi \to Y\) be a submersion. For any submanifold \(A\) of \(Y\), the manifold \(\pi^{-1}(b)\) is transverse to it for generic choices of \(b\).

We shall also need the following crucial observation:

**Lemma 1.1.6.** Let \((\mathbb{P}^2)^r^*\) be a subset of \((\mathbb{P}^2)^r\) with the following property: first of all, the hyperplanes \(H_1, \ldots, H_r\) intersect transversely. Secondly, for all \(i\) the section

\[
\psi_i : H_1 \cap \ldots \cap H_r \to \gamma_b^\perp \otimes \mathbb{C}^2 \quad \text{given by} \quad \psi_i(f) := \nabla f|_{\tilde{p}_i}
\]

is transverse to the zero set. Then the complement of \((\mathbb{P}^2)^r^*\) inside \((\mathbb{P}^2)^r\) is a finite union of varieties of dimension strictly smaller than \(2r\), (i.e., \((\mathbb{P}^2)^r^*\) is a set of full measure), provided \(r < \delta_d\).

**Proof:** Let us assume that a general degree \(d\)-curve is given by

\[
f(X, Y, Z) := f_0Z^d + f_{10}Z^{d-1}X + f_{01}Z^{d-1}Y + f_{20}Z^{d-2}X^2 + \ldots + f_{0d}Y^d.
\]

Let

\[
([X_1 : Y_1 : Z_1], \ldots, [X_r : Y_r : Z_r]) \in (\mathbb{P}^2)^r
\]

be arbitrary choice of \(r\) points. Given these \(r\) points, our desired hypothesis dictates that the following matrix should have full rank:

\[
\begin{pmatrix}
Z_1^d & Z_1^{d-1}X_1 & Z_1^{d-1}Y_1 & Z_1^{d-2}X_1^2 & \ldots & Y_1^d \\
Z_2^d & Z_2^{d-1}X_2 & Z_2^{d-1}Y_2 & Z_2^{d-2}X_2^2 & \ldots & Y_2^d \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
Z_r^d & Z_r^{d-1}X_r & Z_r^{d-1}Y_r & Z_r^{d-2}X_r^2 & \ldots & Y_r^d \\
0 & Z_i^{d-1} & 0 & 2Z_i^{d-2}X_i & \ldots & 0 \\
0 & 0 & Z_i^{d-1} & 0 & \ldots & dY_i^{d-1}
\end{pmatrix}
\]

for all \(i\).

We claim that for each \(i\), the set of points for which the matrix does not have full rank is contained in a proper subvariety of \((\mathbb{P}^2)^r\) of dimension less than \(2r\). This implies the Lemma. It also suffices to establish the claim for \(i = 1\).

Let us rewrite the matrix as

\[
M := \begin{pmatrix}
Z_1^d & Z_1^{d-1}X_1 & Z_1^{d-1}Y_1 & Z_1^{d-2}X_1^2 & \ldots & Y_1^d \\
0 & Z_1^{d-1} & 0 & 2Z_1^{d-2}X_1 & \ldots & 0 \\
0 & 0 & Z_1^{d-1} & 0 & \ldots & dY_1^{d-1} \\
Z_2^d & Z_2^{d-1}X_2 & Z_2^{d-1}Y_2 & Z_2^{d-2}X_2^2 & \ldots & Y_2^d \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
Z_r^d & Z_r^{d-1}X_r & Z_r^{d-1}Y_r & Z_r^{d-2}X_r^2 & \ldots & Y_r^d
\end{pmatrix} = \begin{pmatrix}
M_0 & M_1 \\
M_2 & M_3
\end{pmatrix}.
\]
Since \( r < \delta_d \), let \( \tilde{M} \) be the \((r+2) \times (r+2)\) matrix obtained from \( M \) by taking the first \((r+2)\) columns. In other words, we may write

\[
\tilde{M} := \begin{pmatrix}
M_0 & \tilde{M}_1 \\
\tilde{M}_2 & \tilde{M}_3
\end{pmatrix},
\]

where \( \tilde{M}_i \) are the appropriate truncated version of \( M_i \). If the matrix \( M \) does not have full rank then \( \det \tilde{M} = 0 \). Consider \( \Phi := \det \tilde{M} \) as a function of all the points \( p_i \). Towards showing that this function is not identically zero, it suffices to show that \( \Phi \) is not identically zero when \( X_1 = 0 \) and \( Y_1 = 0 \). Note that \( \tilde{M}_1 \) becomes the zero matrix when \( X_1 = 0 \) and \( Y_1 = 0 \). Hence,

\[
\Phi(X_1 = 0, Y_1 = 0) = \det M_0 \cdot \det \tilde{M}_3.
\]

Note that the \( i^{\text{th}} \) row of \( \det \tilde{M}_3 \) is a function of \((X_i, Y_i, Z_i)\), where \( i = 2, \ldots, r \). Hence, when we take the determinant, it is going to be a non-identically zero function of all the \( \{(X_k, Y_k, Z_k)\}_{k=2}^r \). This proves our claim that \( \Phi \) is not identically zero.

\[\square\]

**Remark 1.1.7.** Lemma 1.1.6 is actually true even when \( r \geq \delta_d \). The intersections in this case are all going to be empty and hence vacuously transverse. However, since we need the Lemma only for \( r < \delta_d \), we stated it in that form.

**Corollary 1.1.8.** Let \((\mathbb{P}^2)^r^*\) be the subset of \((\mathbb{P}^2)^r\) obtained from Lemma 1.1.6. Then the hyperplanes \( \tilde{H}_1, \ldots, \tilde{H}_r \) intersect transversely, provided \( r < \delta_d \). Moreover, for all \( i \) the section

\[
\psi_i : \tilde{H}_1 \cap \ldots \cap \tilde{H}_r \longrightarrow \gamma^*_D \otimes \mathbb{C}^2 \quad \text{given by} \quad \psi_i(\tilde{f}) := \nabla f|_{\tilde{p}_i}
\]

is transverse to the zero set.

Next, let us introduce a new notation:

\[
\begin{align*}
H(r)^* &:= \{(\tilde{f}, \tilde{p}_1, \ldots, \tilde{p}_r) \in D \times (\mathbb{P}^2)^r^* : f(p_1) = 0, \nabla f|_{p_1} \neq 0, \ldots, f(p_r) = 0, \nabla f|_{p_r} \neq 0 \} \\
\hat{H}(r)^* &:= H(r)^* \times \mathbb{P}^2
\end{align*}
\]

(1.1.2)

where \((\mathbb{P}^2)^r^*\) is as defined in Lemma 1.1.6. Note that \( H(r)^* \) is a manifold of dimension \( \delta_d + r \). We shall also need the following:

**Lemma 1.1.9.** The map \( \pi : H(r)^* \longrightarrow (\mathbb{P}^2)^r^* \) is surjective and is a fibre bundle, provided \( r < \delta_d \).

**Proof:** First, we will show surjectivity of the map \( \pi \). Let \((p_1, \ldots, p_r) \in (\mathbb{P}^2)^r^* \). Since \( r < \delta_d \), there exists an \( \tilde{f} \in D \) such that

\[
f(p_1) = 0, \ f(p_2) = 0, \ldots, \ f(p_r) = 0.
\]

(In fact \( r \leq \delta_d \) is all that is required to draw the above conclusion). Let us also assume that

\[
\nabla f|_{p_1} = 0, \ \nabla f|_{p_2} = 0, \ldots, \nabla f|_{p_k} = 0, \ \nabla f|_{p_{k+1}} \neq 0, \ldots, \nabla f|_{p_r} \neq 0.
\]

By the definition of \((\mathbb{P}^2)^r^*\), there exists a curve \( f(t_1, t_2, \ldots, t_k) \) near \( f \) such that

\[
f_t(p_1) = 0, \ f_t(p_2) = 0, \ldots, \ f_t(p_r) = 0
\]

\[
\nabla f_t|_{p_1} = t_1, \ \nabla f_t|_{p_2} = t_2, \ldots, \nabla f_t|_{p_k} = t_k, \ \nabla f_t|_{p_{k+1}} \neq 0, \ldots, \nabla f_t|_{p_r} \neq 0
\]
equations in the variables \( r \) are linearly independent. This means that there is an arbitrary element \( \tilde{t} \) that is invertible. Without loss of generality let us assume its the first element. It is easy to see this is without any loss of generality. We can then identify \( \tilde{t} \) be an arbitrary collection of \( r \) points in \((\mathbb{P}^2)^r\). We can identify each point \( p_i \approx (x_i, y_i) \in \mathbb{C}^2 \) after choosing a coordinate chart. Let us assume an arbitrary element \( f \in D \) is given by

\[
f(X,Y,Z) = f_{00}Z^d + f_{01}Z^{d-1}X + f_{10}Z^{d-1}Y + \ldots + f_{0d}Y^d.
\]

Here \( f_{ij} \) is the coefficient of \( Z^{d-(i+j)}X^iY^j \). Suppose \( \bar{f} \in \pi^{-1}(p_1, \ldots, p_r) \) such that \( f_{00} \neq 0 \) (it is easy to see this is without any loss of generality). We can then identify \( \bar{f} \) with

\[
\rho := (\rho_{01}, \rho_{02}, \ldots, \rho_{0d}) \in \mathbb{C}^{\delta_d} \quad \text{where} \quad \rho_{ij} = \frac{f_{ij}}{f_{00}}.
\]

Hence, in local coordinates, our polynomial is given by

\[
\rho(x,y) = 1 + \rho_{10}x + \rho_{01}y + \ldots \rho_{0d}y^d.
\]

Now, for convenience let us relabel the coefficients \( \rho_{ij} \) as \( g_k \), with only one number. In other words

\[
(g_1, \ldots, g_{\delta_d}) := (\rho_{01}, \rho_{02}, \ldots, \rho_{0d}).
\]

This will make our discussion easier. From the definition of \((\mathbb{P}^2)^r\), we conclude that the \( r \) linear equations in the variables \( \{g_k\}_{k=1}^{\delta_d} \)

\[
\left\{ \rho(x_j, y_j) = 1 + g_1x_j + g_2y_j + \ldots + g_{\delta_d}y_j^d = 0 \right\}_{j=1}^r
\]

are linearly independent. This means that there is an \( r \times r \) minor inside the \( r \times \delta_d \) matrix

\[
\begin{pmatrix}
x_1 & y_1 & x_1^2 & \ldots & y_1^d \\
x_2 & y_2 & x_2^2 & \ldots & y_2^d \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
x_r & y_r & x_r^2 & \ldots & y_r^d
\end{pmatrix},
\]

that is invertible. Without loss of generality let us assume its the first \( r \times r \) minor. Now let \( U \) be an sufficiently small open neighbourhood of \((p_1, \ldots, p_r) \in (\mathbb{P}^2)^r\). Observe that the map

\[
h : \pi^{-1}(U) \rightarrow U \times \mathbb{C}^{\delta_d-r} \quad \text{given by} \quad h(p_1(t), \ldots, p_r(t), \bar{f}) := (p_1(t), \ldots, p_r(t); g_{r+1}(t), \ldots, g_{\delta_d}(t))
\]

is a trivialization, where \( p_i(t) \) and \( g_k(t) \) are points sufficiently close to \( p_i \) and \( g_k \). The map \( h \) is well defined, because the first \( r \times r \) minor of the matrix

\[
\begin{pmatrix}
x_1(t) & y_1(t) & x_1(t)^2 & \ldots & y_1(t)^d \\
x_2(t) & y_2(t) & x_2(t)^2 & \ldots & y_2(t)^d \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
x_r(t) & y_r(t) & x_r(t)^2 & \ldots & y_r(t)^d
\end{pmatrix},
\]

is still invertible if \( t \) is sufficiently small. Hence, \( h \) is a trivialization and \( \pi : H \rightarrow (\mathbb{P}^2)^r \) is a fibre bundle. \( \square \)
Lemma 1.1.10. The projection map $\pi_D : \mathcal{H}(r)^* \to \mathcal{D}$ is a submersion if $r < \delta_d$.

Proof: Follows from the setup of Lemma 1.1.9, by writing the map $\pi_D$ in local coordinates. \qed

Corollary 1.1.11. The projection map $\pi_D : \hat{\mathcal{H}}(r)^* \to \mathcal{D} \times \mathbb{P}^2$ is a submersion and $\pi : \hat{\mathcal{H}}(r)^* \to (\mathbb{P}^2)^{\ast}$ is a fiber bundle if $r < \delta_d$.

Proof: Follows immediately from Lemma 1.1.9 and 1.1.10. \qed

Lemma 1.1.12. For generic choices of $\delta_d - k$ points in $(\mathbb{P}^2)^{(\delta_d - k)^{\ast}}$, the intersection

$$\mathcal{X}_k \cap \hat{\mathcal{H}}_1^* \cap \hat{\mathcal{H}}_2^* \ldots \cap \hat{\mathcal{H}}_{\delta_d - k}^*$$

is transverse, provided $d \geq C_{\mathcal{X}_k}$ and $k \geq 1$.

Proof: Follows immediately from Corollary 1.1.5 and 1.1.11. The bound $d \geq C_{\mathcal{X}_k}$ is imposed to ensure $\mathcal{X}_k$ is a smooth manifold. \qed

At this point we shall require a technical fact which will be useful in proving density of spaces.

Lemma 1.1.13. Let $\rho_0 : \mathbb{C}^2 \to \mathbb{C}$ be a non zero polynomial of degree at most $d$. Suppose $(x_0, y_0) \in \mathbb{C}^2$ is a given point such that $\rho_0(x_0, y_0) = 0$. Then given any other polynomial $\rho_t : \mathbb{C}^2 \to \mathbb{C}$ of degree at most $d$ which is sufficiently close to $\rho_0$, there exists a point $(x_t, y_t)$, not necessarily unique, close to $(x_0, y_0)$ such that $\rho_t(x_t, y_t) = 0$. Furthermore, the roots $(x_t, y_t)$ converge to $(x_0, y_0)$.

Proof: Without loss of generality we can assume $(x_0, y_0) = (0, 0)$. After a linear change of coordinates, we can assume that

$$\rho(x, 0) = x^k g(x)$$

where $g(0) \neq 0$. Choose a small disk $D$ around 0 so that $\rho(x, 0)$ has no zeros inside $D - \{0\}$. Choose $\rho_t$ sufficiently close to $\rho$ so that $\rho_t(x, 0)$ does not vanish on $\partial D$. Hence,

$$n_t := \frac{1}{2\pi i} \oint_{\partial D} \frac{\partial \rho_t(x, 0)}{\rho_t(x, 0)} dx$$

is a continuous function of $t$. But this is the number of roots of $\rho_t(x, 0)$ inside $D$ and is, therefore, a constant. Hence, $\rho_t(x, 0)$ has $k$ roots inside $D$. Moreover, these roots tend to zero as $t$ tends to zero. This proves the claim. \qed

Let us introduce a couple of more notations. Given a singularity of type $\mathcal{X}_k$ and a point $\tilde{p} \in \mathbb{P}^2$, we define

$$\mathcal{X}_k(\tilde{p}) := \{ \tilde{f} \in \mathcal{D} : \tilde{f} \text{ has a singularity of type } \mathcal{X}_k \text{ at } \tilde{p} \}$$

$$\mathcal{X}_k^*(\tilde{p}) := \{ \tilde{f} \in \mathcal{D} : \tilde{f} \text{ has a singularity of type } \mathcal{X}_k \text{ at } \tilde{p} \text{ and no other singular point} \}.$$  \hspace{1cm} (1.1.3)

Lemma 1.1.14. The space $\mathcal{X}_k^*(\tilde{p})$ is open and dense in $\mathcal{X}_k(\tilde{p})$ for all $\tilde{p}$ if $d \geq C_{\mathcal{X}_k}$.

Proof: It is equivalent to showing that $\mathcal{X}_k^*(\tilde{p})$ is open and dense inside $\overline{\mathcal{X}}_k(\tilde{p})$, the closure of $\mathcal{X}_k(\tilde{p})$ inside $\mathcal{D}$. The family of curves that belongs to $\overline{\mathcal{X}}_k(\tilde{p})$ forms a linear system. We claim that the base locus of this family is $\tilde{p}$. Assuming this, we are done, because by Bertini’s theorem a generic element of $\overline{\mathcal{X}}_k(\tilde{p})$ is smooth away from the base locus. Hence, $\mathcal{X}_k^*(\tilde{p})$ is open and dense inside $\overline{\mathcal{X}}_k(\tilde{p})$. \hspace{1cm} (1.1.0)
In order to prove that the base locus is $\tilde{p}$, let us first assume $X_k = A_k$. Without loss of generality, we can assume that $\tilde{p} = [0 : 0 : 1]$. Consider the family of curves

$$y^2z^{d-3} + x^{k+1}z^{d-(k+1)} = 0, \quad y^2z^{d-3} + x^kz^{d-(k+1)} + x^d = 0, \quad y^2z^{d-3} + x^kz^{d-(k+1)} + y^d = 0.$$ 

These have an $A_k$-node at $[0 : 0 : 1]$ and their common zero is $[0 : 0 : 1]$. This argument works, provided $d \geq \max(2, k + 1) = C_{A_k}$, which proves the claim for $A_k$.

Now let $X_k = D_k$. Consider the family of curves

$$xy^2z^{d-3} + x^{k-1}z^{d-(k-1)} = 0, \quad xy^2z^{d-3} + x^{k-1}z^{d-(k-1)} + x^d = 0, \quad xy^2z^{d-3} + x^{k-1}z^{d-(k-1)} + y^d = 0.$$ 

It is obvious that the first and second curves have a $D_k$-node at $[0 : 0 : 1]$. To see why the third curve has a $D_k$-node, we can simply use Lemma 1.4.11. The base locus of this family is $[0 : 0 : 1]$. This argument works, provided $d \geq \max(3, k - 1) = C_{D_k}$, which proves the claim for $D_k$.

For $E_6$, $E_7$ we can consider the family of curves

$$y^3z^{d-3} + x^4z^{d-4} = 0, \quad y^3z^{d-3} + x^4z^{d-4} + x^d = 0, \quad y^3z^{d-3} + x^4z^{d-4} + y^d = 0$$

and

$$y^3z^{d-3} + yx^3z^{d-4} = 0, \quad y^3z^{d-3} + yx^3z^{d-4} + y^d = 0, \quad y^3z^{d-3} + yx^3z^{d-4} + x^d = 0$$

respectively. To see why the last curve has an $E_7$-node, apply Lemma 1.4.16. This argument works provided $d \geq 4 = C_{E_6} = C_{E_7}$. This completes the proof of Lemma 1.1.14.

Let us now define the following $r$-fold fibered products:

$$\mathcal{C}(r) := \{(\tilde{f}, \tilde{p}, \tilde{p}_1, \ldots, \tilde{p}_r) \in X_k \times (\mathbb{P}^2)^r : f(p_1) = 0, \ldots, f(p_r) = 0\}$$

$$C(r) := \{(\tilde{f}, \tilde{p}, \tilde{p}_1, \ldots, \tilde{p}_r) \in X_k \times (\mathbb{P}^2)^r : f(p_1) = 0, \ldots, f(p_r) = 0, \nabla f|_{p_1} \neq 0 \ldots, \nabla f|_{p_r} \neq 0\}$$

(1.1.4)

Note that a priori, $\mathcal{C}(r)$ may not be the closure of $C(r)$. The next Lemma shows that we have chosen a good notation.

**Lemma 1.1.15.** The space $C(r)$ is open and dense in $\mathcal{C}(r)$ if $d \geq C_{X_k}$ and $k \geq 1$.

**Proof:** The set is obviously open. To show that it is dense, let us assume that $(\tilde{f}, \tilde{p}, \tilde{p}_1, \ldots, \tilde{p}_r) \in \mathcal{C}(r)$. We need to show that there exists a sequence in $C(r)$, that converges to $(\tilde{f}, \tilde{p}, \tilde{p}_1, \ldots, \tilde{p}_r)$. First of all, since $(\tilde{f}, \tilde{p}) \in X_k$, there exists a sequence $(\tilde{f}_{n_1}, \tilde{p}(n_1))$ in $X_k$ that converges to $(\tilde{f}, \tilde{p})$. By Lemma 1.1.14, there exists a sequence $(\tilde{f}_{n_1, n_2})$ converging to $\tilde{f}_{n_1}$, which has only $\tilde{p}(n_1)$ as a singular point. By Lemma 1.1.13 there exists a sequence $(\tilde{p}_1(n_1, n_2), \ldots, \tilde{p}_r(n_1, n_2))$ converging to $(\tilde{p}_1, \ldots, \tilde{p}_r)$ that lies on the curve $\tilde{f}_{n_1, n_2}$. Since $\tilde{f}_{n_1, n_2}$ has only one singular point, there exists a sequence $(\tilde{p}_1(n_1, n_2, n_3), \ldots, \tilde{p}_r(n_1, n_2, n_3))$ converging to $(\tilde{p}_1(n_1, n_2), \ldots, \tilde{p}_r(n_1, n_2))$ that are smooth points of the curve. Hence, $(\tilde{f}_{n_1, n_2}, \tilde{p}(n_1, n_2, n_3), \tilde{p}_1(n_1, n_2, n_3), \ldots, \tilde{p}_r(n_1, n_2, n_3)) \in C(r)$ is the desired sequence.

**Lemma 1.1.16.** For generic $(p_1, \ldots, p_{d-k}) \in (\mathbb{P}^2)^{d-k}$, we have

$$X_k \cap \hat{H}_1 \cap \ldots \cap \hat{H}_{d-k} = X_k \cap \hat{H}_1^* \cap \ldots \cap \hat{H}_{d-k}^*,$$

if $d \geq C_{X_k}$ and $k \geq 1$. 


Proof: Let \( \pi_2 : \mathcal{C}(r) \rightarrow (\mathbb{P}^2)^r \) be the projection onto the second factor. It is easy to see that if \( (p_1, \ldots, p_r) \in (\mathbb{P}^2)^r - \pi_2(\mathcal{C}(r) - \mathcal{C}(r)) \) then

\[
\mathcal{X}_k \cap \hat{H}_1 \ldots \cap \hat{H}_r = \mathcal{X}_k \cap \hat{H}_1^* \ldots \cap \hat{H}_r^*.
\]

Hence, all we need to show is that if \( r = \delta_d - k \) (which is the dimension of \( \mathcal{X}_k \)), then \( \pi_2(\mathcal{C}(r) - \mathcal{C}(r)) \) is contained in a union of varieties of dimension less than \( 2r \). If \( r \) is the dimension of \( \mathcal{X}_k \), then the dimension of \( \mathcal{C}(r) \) and \( \mathcal{C}(r) \) is \( 2r \) (the dimension as an algebraic variety). By Lemma 1.1.15, the complement \( \mathcal{C}(r) - \mathcal{C}(r) \) is contained in a union of varieties of dimension less than \( 2r \). Hence, \( \pi_2(\mathcal{C}(r) - \mathcal{C}(r)) \) is contained in a union of varieties of dimension less then \( 2r \), which proves the claim. \( \square \)

Proof of Lemma 1.1.1: Lemma 1.1.16 and 1.1.12 imply Lemma 2.6 in [1] (cf. Lemma 1.1.1 here), for \( n = 0 \). Similar reasoning holds for \( n = 1 \) and \( 2 \). \( \square \)

1.2 Relationship between \( \mathcal{N}(\mathcal{X}_k, n) \) and \( \mathcal{N}(\mathcal{P}\mathcal{X}_k, n, 0) \)

Lemma 1.2.1. The projection map \( \pi : \mathcal{P}\mathcal{X}_k \rightarrow \mathcal{X}_k \) is one to one if \( \mathcal{X}_k = A_k, D_k, E_6, E_7 \) or \( E_8 \) except for \( \mathcal{X}_k = D_4 \) when it is three to one. In particular,

\[
\mathcal{N}(\mathcal{X}_k, n) = \mathcal{N}(\mathcal{P}\mathcal{X}_k, n, 0) \quad \text{if} \quad \mathcal{X}_k \neq D_4 \quad \text{and} \quad \mathcal{N}(D_4, n) = \frac{\mathcal{N}(\mathcal{P}D_4, n, 0)}{3}. \tag{1.2.1}
\]

Proof: Let \( \mathcal{X}_k = A_k \). If \( (\tilde{f}, \tilde{l}_p) \in \mathcal{P}A_k \) then the projection map has to be one to one, because otherwise the kernel of the Hessian would have two linearly independent vectors (which would imply it is identically zero). Similar argument holds if \( \mathcal{X}_k = D_k \) for \( k \geq 5 \) or \( E_6, E_7 \) or \( E_8 \). In all of the above cases, the map \( \mathcal{P}\mathcal{X}_k \rightarrow \mathcal{X}_k \) is a diffeomorphism (actually, a biholomorphism). Since \( \mathcal{X}_k \) and \( \pi(\mathcal{P}\mathcal{X}_k) \) are cobordant, the pseudocycles \([\mathcal{X}_k]\) and \( \pi_*(\mathcal{P}\mathcal{X}_k) \), defined by \( \mathcal{X}_k \) and \( \pi(\mathcal{P}\mathcal{X}_k) \) respectively, define the same homology class. This proves (1.2.1).

If \( (\tilde{f}, \tilde{p}) \in D_4 \) then there exists three distinct directions \( l_\tilde{p} \) along which the third derivative vanishes. Hence, the projection map is three to one and orientation preserving. If we count the signed intersection number then we see that each transverse intersection point in \( D_4 \) is accounted for thrice (with the same sign) when counted in \( \mathcal{P}D_4 \). This proves (1.2.1) for \( \mathcal{X}_k = D_4 \). We note that this method of counting transverse intersection points also work identically for the first part above where the map \( \mathcal{P}\mathcal{X}_k \rightarrow \mathcal{X}_k \) is one to one. \( \square \)

1.3 Summary of definitions and notation

1.3.1 The vector bundles involved

We now list down all the vector bundles that we will encounter in this manuscript. Let \( \gamma_{\mathbb{P}^2} \rightarrow \mathbb{P}^2, \gamma_D \rightarrow D \) and \( \tilde{\gamma} \rightarrow \mathbb{P}\mathbb{T}\mathbb{P}^2 \) be the tautological line bundles over \( \mathbb{P}^2, D \) and \( \mathbb{P}\mathbb{T}\mathbb{P}^2 \) respectively and \( \pi : D \times \mathbb{P}\mathbb{T}\mathbb{P}^2 \rightarrow D \times \mathbb{P}^2 \) be the projection map. We have the following bundles
over $D \times \mathbb{P}^2$:

$$\mathcal{L}_{A_0} := \gamma_D \otimes \gamma_{p_2}^{*d} \to D \times \mathbb{P}^2$$
$$\mathcal{V}_{A_1} := \gamma_D ^{*} \otimes \gamma_{p_2}^{*d} \otimes T^*\mathbb{P}^2 \to D \times \mathbb{P}^2$$
$$\mathcal{L}_{A_2} := (\gamma_D ^{*d} \otimes \gamma_{p_2}^{*d} \otimes \Lambda^2 T^*\mathbb{P}^2)^{\otimes 2} \to D \times \mathbb{P}^2$$
$$\mathcal{V}_{D_1} := \gamma_D ^{*} \otimes \gamma_{p_2}^{*d} \otimes \text{Sym}^2 (T^*\mathbb{P}^2 \otimes T^*\mathbb{P}^2) \to D \times \mathbb{P}^2$$
$$\mathcal{V}_{A_3} := \gamma_D ^{*} \otimes \gamma_{p_2}^{*d} \otimes \text{Sym}^3 (T^*\mathbb{P}^2 \otimes T^*\mathbb{P}^2 \otimes T^*\mathbb{P}^2) \to D \times \mathbb{P}^2$$

Associated to the map $\pi$ there are pullback bundles

$$L_{A_0} := \pi^* L_{A_0} \to D \times \mathbb{P}^2 T^2$$
$$V_{A_1} := \pi^* \mathcal{V}_{A_1} \to D \times \mathbb{P}^2 T^2$$
$$V_{D_1} := \pi^* \mathcal{V}_{D_1} \to D \times \mathbb{P}^2 T^2$$
$$V_{A_3} := \pi^* \mathcal{V}_{A_3} \to D \times \mathbb{P}^2 T^2$$

Finally, we have

$$L_{PD_1} := (T^*\mathbb{P}^2 / \tilde{\gamma})^2 \otimes \gamma_D ^{*} \otimes \gamma_{p_2}^{*d} \to D \times \mathbb{P}^2 T^2$$
$$L_{PD_5} := \tilde{\gamma}^* \otimes (T^*\mathbb{P}^2 / \tilde{\gamma})^* \otimes \gamma_D ^{*} \otimes \gamma_{p_2}^{*d} \to D \times \mathbb{P}^2 T^2$$
$$L_{PD_6} := \tilde{\gamma}^* \otimes (T^*\mathbb{P}^2 / \tilde{\gamma})^* \otimes \gamma_D ^{*d} \otimes \gamma_{p_2}^{*d} \to D \times \mathbb{P}^2 T^2$$
$$L_{P_3} := \tilde{\gamma}^* \otimes (T^*\mathbb{P}^2 / \tilde{\gamma})^* \otimes \gamma_D ^{*d} \otimes \gamma_{p_2}^{*d} \to D \times \mathbb{P}^2 T^2$$
$$L_{PD_5} := \tilde{\gamma}^* \otimes (T^*\mathbb{P}^2 / \tilde{\gamma})^* \otimes \gamma_D ^{*d} \otimes \gamma_{p_2}^{*3} \to D \times \mathbb{P}^2 T^2$$

$$k \geq 3$$
$$L_{PD_5} := \tilde{\gamma}^{*k} \otimes (T^2 / \tilde{\gamma})^{*(2k-6)} \otimes \gamma_D ^{*(k-2)} \otimes \gamma_{p_2}^{*(d(k+1)-3d)} \to D \times \mathbb{P}^2 T^2$$

$$k \geq 6$$
$$L_{PD_5} := \tilde{\gamma}^{*(k-2+\epsilon_k)} \otimes (T^2 / \tilde{\gamma})^{*(2k)} \otimes \gamma_D ^{*(1+\epsilon_k)} \otimes \gamma_{p_2}^{*(d(1+\epsilon_k))} \to D \times \mathbb{P}^2 T^2,$$

where $\epsilon_0 = 0$, $\epsilon_7 = 1$ and $\epsilon_8 = 3$. In general, $\epsilon_k$ is the order of the pole of the section $D_k^f$ at $f_{12} = 0$. The algorithm to obtain $D_k^f$ for any $k$ is given in Lemma 1.4.11. The reason for defining these bundles will become clearer in the subsection 1.3.2, when we define sections of these bundles.

Recall that we are making an abuse of notation by omitting to write the pullback maps in our notation. The bundle $T^2 / \tilde{\gamma}$ is the quotient of the bundles $V$ and $W$, where $V$ is the pullback of the tangent bundle $T^2 \to \mathbb{P}^2$ via $D \times \mathbb{P}^2 \overset{\pi}{\to} D \times \mathbb{P}^2 \to \mathbb{P}^2$ and $W$ is pullback of $\tilde{\gamma} \to \mathbb{P}^2 T^2$ via $D \times \mathbb{P}^2 \to \mathbb{P}^2 T^2$. 

11
1.3.2 Sections of Vector Bundles

Let us define the notion of vertical derivatives.

**Definition 1.3.1.** Let $\pi : V \rightarrow M$ be a holomorphic vector bundle of rank $k$ and $s : M \rightarrow V$ be a holomorphic section. Suppose $h : V|_U \rightarrow U \times \mathbb{C}^k$ is a holomorphic trivialization of $V$ and $\pi_1, \pi_2 : U \times \mathbb{C}^k \rightarrow U, \mathbb{C}^k$ the projection maps. Let

$$\tilde{s} := \pi_2 \circ h \circ s.$$  \hspace{1cm} (1.3.1)

For $q \in U$, we define the vertical derivative of $s$ to be the $\mathbb{C}$-linear map

$$\nabla s|_q : T_qM \rightarrow V_q, \quad \nabla s|_q := (\pi_2 \circ h)|^{-1}_{V_q} \circ d\tilde{s}|_q,$$

where $V_q = \pi^{-1}(q)$, the fibre at $q$. In particular, if $v \in T_qM$ is given by a holomorphic map $\gamma : B_\epsilon(0) \rightarrow M$ such that $\gamma(0) = q$ and $\frac{\partial \gamma}{\partial z}|_{z=0} = v$, then

$$\nabla s|_q(v) := (\pi_2 \circ h)|^{-1}_{V_q} \circ \frac{\partial \tilde{s}(z)}{\partial z}|_{z=0}$$

were $B_\epsilon$ is an open $\epsilon$-ball in $\mathbb{C}$ around the origin.\(^2\) Finally, if $v, w \in T_qM$ are tangent vectors such that there exists a family of complex curves $\gamma : B_\epsilon \times B_\epsilon \rightarrow M$ such that

$$\gamma(0, 0) = q, \quad \frac{\partial \gamma(x, y)}{\partial x}|_{(0,0)} = v, \quad \frac{\partial \gamma(x, y)}{\partial y}|_{(0,0)} = w$$

then

$$\nabla^{i+j}s|_q(v, \ldots, v, w, \ldots, w) := (\pi_2 \circ h)|^{-1}_{V_q} \circ \left[ \frac{\partial^{i+j} \tilde{s}(\gamma(x, y))}{\partial x^i \partial y^j} \right]|_{(0,0)}.$$  \hspace{1cm} (1.3.2)

**Remark 1.3.2.** In general the quantity in (2.3.2) is not well defined, i.e., it depends on the trivialization and the curve $\gamma$. Lemma 1.4.18 explains on what subspace this quantity is well defined.

**Remark 1.3.3.** The section $s : M \rightarrow V$ is transverse to the zero set if and only if the induced map

$$\tilde{s} := \tilde{s} \circ \varphi_U^{-1} : \mathbb{C}^m \rightarrow \mathbb{C}^k$$  \hspace{1cm} (1.3.3)

is transverse to the zero set in the usual calculus sense, where $\varphi_U : U \rightarrow \mathbb{C}^m$ is a coordinate chart and $\tilde{s}$ is as defined in (2.3.1).

Let $f : \mathbb{P}^2 \rightarrow \gamma^{\mathbb{C}}_{\mathbb{C}}$ be a section and $\tilde{p} \in \mathbb{P}^2$. We can think of $p$ as a non-zero vector in $\gamma_{\mathbb{R}^2}$ and $p^{\mathbb{C}}$ a non-zero vector in $\gamma^{\mathbb{C}}_{\mathbb{R}^2}$ \(^3\). The quantity $\nabla f|_{\tilde{p}}$ acts on a vector in $\gamma_{\mathbb{R}^2}^{\mathbb{C}}|_{\tilde{p}}$ and produces an element of $T_{\tilde{p}}^{\mathbb{C}}\mathbb{P}^2$. Let us denote this quantity as $\nabla f|_{\tilde{p}}$, i.e.,

$$\nabla f|_{\tilde{p}} := \{\nabla f|_{\tilde{p}}\}(p^{\mathbb{C}}) \in T_{\tilde{p}}^{\mathbb{C}}\mathbb{P}^2.$$  \hspace{1cm} (1.3.4)

\(^2\)Not every tangent vector is given by a holomorphic map; however combined with the fact that $\nabla s|_p$ is $\mathbb{C}$-linear, this definition determines $\nabla s|_p$ completely.

\(^3\)Remember that $p$ is an element of $\mathbb{C}^3$ – 0 while $\tilde{p}$ is the corresponding equivalence class in $\mathbb{P}^2$.  


Notice that $\nabla f|_{\tilde{p}}$ is an element of the fibre of $T^*\mathbb{P}^2 \otimes \gamma_{\tilde{p}}$ at $\tilde{p}$ while $\nabla f|_{p}$ is an element of $T^*_p\mathbb{P}^2$.

Now observe that $\pi^*\mathbb{P}^2 \cong \tilde{\gamma} \oplus \pi^*\mathbb{P}^2/\tilde{\gamma} \longrightarrow \mathbb{P}T\mathbb{P}^2$, where $\pi : \mathbb{P}T\mathbb{P}^2 \longrightarrow \mathbb{P}^2$ is the projection map. Let us denote a vector in $\tilde{\gamma}$ by $v$ and a vector in $\pi^*\mathbb{P}^2/\tilde{\gamma}$ by $\tilde{w}$. Given $\tilde{f} \in \mathcal{D}$ and $p \in \mathbb{P}^2$, let

$$f_{ij} := \nabla^{i+j} f|_p(v, \ldots, v, w, \ldots w). \quad (1.3.5)$$

Note that $f_{ij}$ is a number. In general $f_{ij}$ is not well defined; it depends on the trivialization and the curve. Moreover it is also not well defined on the quotient space. Since our sections are not defined on the whole space, we will use the notation $s : M \longrightarrow V$ to indicate that $s$ is defined only on a subspace of $M$. With this terminology, we now explicitly define the sections that we will encounter in this paper.

$$\psi_{A_0} : \mathcal{D} \times \mathbb{P}^2 \longrightarrow \mathcal{L}_{A_0}, \quad \{\psi_{A_0}(\tilde{f}, \tilde{p})\}(f \otimes p^\otimes d) := f(p)$$
$$\psi_{A_1} : \mathcal{D} \times \mathbb{P}^2 \longrightarrow \mathcal{V}_{A_1}, \quad \{\psi_{A_1}(\tilde{f}, \tilde{p})\}(f \otimes p^\otimes d) := \nabla f|_p$$
$$\psi_{D_1} : \mathcal{D} \times \mathbb{P}^2 \rightarrow \mathcal{V}_{D_1}, \quad \{\psi_{D_1}(\tilde{f}, \tilde{p})\}(f \otimes p^\otimes d) := \nabla^2 f|_p$$
$$\psi_{X_5} : \mathcal{D} \times \mathbb{P}^2 \rightarrow \mathcal{V}_{X_5}, \quad \{\psi_{X_5}(\tilde{f}, \tilde{p})\}(f \otimes p^\otimes d) := \nabla^3 f|_p$$
$$\psi_{A_2} : \mathcal{D} \times \mathbb{P}^2 \rightarrow \mathcal{L}_{A_2}, \quad \{\psi_{A_2}(\tilde{f}, \tilde{p})\}(f \otimes p^\otimes d) := \det \nabla^2 f|_p$$
$$\Psi_{A_0} : \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 \longrightarrow \mathcal{L}_{A_0}, \quad \Psi_{A_0}(\tilde{f}, \tilde{l}_p) := \psi_{A_0}(\tilde{f}, \tilde{p})$$
$$\Psi_{A_1} : \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 \rightarrow \mathcal{V}_{A_1}, \quad \Psi_{A_1}(\tilde{f}, \tilde{l}_p) := \psi_{A_1}(\tilde{f}, \tilde{p})$$
$$\Psi_{D_4} : \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 \rightarrow \mathcal{V}_{D_4}, \quad \Psi_{D_4}(\tilde{f}, \tilde{l}_p) := \psi_{D_4}(\tilde{f}, \tilde{p})$$
$$\Psi_{X_8} : \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 \rightarrow \mathcal{V}_{X_8}, \quad \Psi_{X_8}(\tilde{f}, \tilde{l}_p) := \psi_{X_8}(\tilde{f}, \tilde{p}).$$

We also have

$$\Psi_{P_{A_2}} : \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 \longrightarrow \mathcal{V}_{P_{A_2}}, \quad \{\Psi_{P_{A_2}}(\tilde{f}, \tilde{l}_p)\}(f \otimes p^\otimes d \otimes v) := \nabla^2 f|_p(v, \cdot)$$
$$\Psi_{P_{D_5}} : \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 \rightarrow \mathcal{V}_{P_{D_5}}, \quad \{\Psi_{P_{D_5}}(\tilde{f}, \tilde{l}_p)\}(f \otimes p^\otimes d \otimes v^\otimes 2) := \nabla^3 f|_p(v, v, \cdot)$$
$$\Psi_{P_{D_4}} : \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 \rightarrow \mathcal{L}_{P_{D_4}}, \quad \{\Psi_{P_{D_4}}(\tilde{f}, \tilde{l}_p)\}(f \otimes p^\otimes d \otimes w^\otimes 2) := f_{02}$$
$$\Psi_{P_{D_5}} : \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 \rightarrow \mathcal{L}_{P_{D_5}}, \quad \{\Psi_{P_{D_5}}(\tilde{f}, \tilde{l}_p)\}(f \otimes p^\otimes d \otimes v^\otimes 2 \odot w) := f_{21}$$
$$\Psi_{P_{E_6}} : \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 \longrightarrow \mathcal{L}_{P_{E_6}}, \quad \{\Psi_{P_{E_6}}(\tilde{f}, \tilde{l}_p)\}(f \otimes p^\otimes d \otimes v \otimes w^\otimes 2) := f_{12}$$
$$\Psi_{P_{E_7}} : \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 \rightarrow \mathcal{L}_{P_{E_7}}, \quad \{\Psi_{P_{E_7}}(\tilde{f}, \tilde{l}_p)\}(f \otimes p^\otimes d \otimes v^\otimes 4) := f_{40}$$
$$\Psi_{P_{E_8}} : \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 \longrightarrow \mathcal{L}_{P_{E_8}}, \quad \{\Psi_{P_{E_8}}(\tilde{f}, \tilde{l}_p)\}(f \otimes p^\otimes d \otimes v^\otimes 3 \otimes w) := f_{31}$$
$$\Psi_{P_{A_8}} : \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 \rightarrow \mathcal{L}_{P_{A_8}}, \quad \{\Psi_{P_{A_8}}(\tilde{f}, \tilde{l}_p)\}(f \otimes p^\otimes d \otimes v^\otimes 3) := f_{03}.$$

We also have sections of the following bundles: $\Psi_{P_{D_5}} : \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 \longrightarrow \mathcal{L}_{P_{D_5}}$ given by

$$\{\Psi_{P_{D_5}}(\tilde{f}, \tilde{l}_p)\}(f^\otimes 2 \otimes p^\otimes d \otimes v^\otimes 2 \otimes w^\otimes 2) := 3f_{12}^2 - 4f_{21}f_{03}, \quad (1.3.6)$$

and $\Psi_{P_{D_5}} : \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 \rightarrow \mathcal{L}_{P_{D_5}}$ given by

$$\{\Psi_{P_{D_5}}(\tilde{f}, \tilde{l}_p)\}(f^\otimes 5 \otimes p^\otimes 5d \otimes v^\otimes 8 \otimes w^\otimes 4) := \left(f_{12}f_{40} - 8f_{12}f_{21}f_{31} + 24f_{12}f_{21}f_{22} - 32f_{12}f_{21}f_{13} + 16f_{13}f_{04}\right), \quad (1.3.7)$$

and $\Psi_{\mathcal{J}} : \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 \longrightarrow \mathcal{L}_{\mathcal{J}}$ given by

$$\{\Psi_{\mathcal{J}}(\tilde{f}, \tilde{l}_p)\}(f^\otimes 3 \otimes p^\otimes 3d \otimes v^\otimes 9 \otimes w^\otimes 3) := \left(-\frac{f_{31}^3}{8} + \frac{3f_{22}f_{31}f_{40}}{16} - \frac{f_{13}f_{40}^2}{16}\right). \quad (1.3.8)$$
When $k \geq 3$ we have $\Psi_{P, A_k} : \mathcal{D} \times \mathbb{P}^2 \to \mathbb{L}_{P, A_k}$ given by
\[
\{\Psi_{P, A_k}(f, l_p)\}(f^\otimes(k-2) \otimes p^\otimes(d(k-2) \otimes v^\otimes k) \otimes w^\otimes(2k-6)) := f_{02}^{k-3} A_k^f.
\]
Similarly, when $k \geq 6$ we have $\Psi_{P, D_k} : \mathcal{D} \times \mathbb{P}^2 \to \mathbb{L}_{P, D_k}$ given by
\[
\{\Psi_{P, D_k}(f, l_p)\}(f^\otimes(1+\epsilon_k) \otimes p^\otimes(1+\epsilon_k) \otimes v^\otimes(k-2+\epsilon_k) \otimes w^\otimes(2\epsilon_k)) := f_{12}^k D_k^f,
\]
where, as seen previously, $\epsilon_k$ is the order of the pole of $f_{12} = 0$ for the section $D_k^f$. In particular, $\epsilon_6 = 0$, $\epsilon_7 = 1$ and $\epsilon_8 = 3$. The expressions for $A_k^f$ (resp. $D_k^f$) are given below explicitly in (1.3.9) (resp. (1.3.10)), till $k = 7$ (resp. till $k = 8$). The algorithm to obtain $A_k^f$ (resp. $D_k^f$) for any $k$ is given in Lemma 1.4.5 (resp. Lemma 1.4.11).

Here is an explicit formula for $A_k^f$ till $k = 7$.

\[
\begin{align*}
A_3^f &= f_{30}, \quad A_4^f = f_{40} - \frac{3f_{21}}{f_{02}}, \quad A_5^f = f_{50} - \frac{10f_{21}f_{31}}{f_{02}^2} + \frac{15f_{12}f_{21}}{f_{02}^3}, \\
A_6^f &= f_{60} - \frac{15f_{21}f_{41}}{f_{02}^3} - \frac{10f_{21}^2}{f_{02}^2} + \frac{60f_{12}f_{21}f_{31}}{f_{02}^2} + \frac{45f_{21}f_{22}}{f_{02}^3} - \frac{15f_{03}f_{21}^3}{f_{02}^4} - \frac{90f_{12}f_{21}^2}{f_{02}^4}, \\
A_7^f &= f_{70} - \frac{21f_{21}f_{51}}{f_{02}^4} - \frac{35f_{31}f_{41}}{f_{02}^3} + \frac{105f_{12}f_{21}f_{41}}{f_{02}^3} + \frac{105f_{21}f_{22}f_{31}}{f_{02}^4} + \frac{70f_{12}f_{31}}{f_{02}^4} + \frac{210f_{21}f_{22}f_{31}}{f_{02}^4} \\
&\quad - \frac{105f_{03}f_{21}^2f_{31}}{f_{02}^5} - \frac{420f_{12}f_{21}f_{31}}{f_{02}^5} + \frac{630f_{12}f_{22}f_{31}}{f_{02}^5} - \frac{105f_{13}f_{21}f_{31}}{f_{02}^5} + \frac{315f_{03}f_{12}f_{21}^3}{f_{02}^6} + \frac{630f_{12}f_{21}^2}{f_{02}^6}.
\end{align*}
\]

Here is an explicit formula for $D_k^f$ till $k = 8$.

\[
\begin{align*}
D_6^f &= f_{40}, \quad D_7^f = f_{50} - \frac{5f_{31}f_{41}}{3f_{12}}, \quad D_8^f = f_{60} + \frac{5f_{03}f_{31}f_{50}}{3f_{12}^2} - \frac{5f_{31}f_{41}}{f_{12}} - \frac{10f_{03}f_{31}^3}{3f_{12}^3} + \frac{5f_{22}f_{21}^2}{f_{12}^2}.
\end{align*}
\]
1.3.3 The spaces involved.

We begin by explaining a terminology. If \( l_\tilde{p} \in \mathbb{P}T_{\tilde{p}}\mathbb{P}^2 \), then we say that \( v \in l_\tilde{p} \) if \( v \) is a tangent vector in \( T_{\tilde{p}}\mathbb{P}^2 \) and lies over the fibre of \( l_\tilde{p} \). We now define the spaces that we will encounter.

\[
\mathcal{X}_k := \{(\tilde{f}, \tilde{p}) \in \mathcal{D} \times \mathbb{P}^2 : f \text{ has a singularity of type } \mathcal{X}_k \text{ at } \tilde{p}\}
\]

\[
\hat{\mathcal{X}}_k := \{(\tilde{f}, \tilde{p}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 : f \text{ has a singularity of type } \mathcal{X}_k \text{ at } \tilde{p}\} = \pi^{-1}(\mathcal{X}_k)
\]

if \( k > 1 \)

\[
\mathcal{P}_k := \{(\tilde{f}, \tilde{p}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 : f \text{ has a singularity of type } \mathcal{A}_k \text{ at } \tilde{p}, \nabla^2 f|_p(v, \cdot) = 0 \text{ if } v \in l_\tilde{p}\}
\]

\[
\mathcal{P}_k := \{(\tilde{f}, \tilde{p}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 : f \text{ has a singularity of type } \mathcal{D}_4 \text{ at } \tilde{p}, \nabla^3 f|_p(v, v, v) = 0 \text{ if } v \in l_\tilde{p}\}
\]

if \( k > 4 \)

\[
\mathcal{P}_k := \{(\tilde{f}, \tilde{p}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 : f \text{ has a singularity of type } \mathcal{E}_k \text{ at } \tilde{p}, \nabla^3 f|_p(v, v, \cdot) = 0 \text{ if } v \in l_\tilde{p}\}
\]

if \( k = 6, 7 \) or \( 8 \)

\[
\mathcal{P}_k := \{(\tilde{f}, \tilde{p}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 : f \text{ has a singularity of type } \mathcal{D}_k \text{ at } \tilde{p}, \nabla^3 f|_p(v, v, v) = 0, \nabla^3 f|_p(v, v, w) \neq 0, \text{ if } v \in l_\tilde{p}, w \in \frac{T_{\tilde{p}}\mathbb{P}^2}{l_\tilde{p}}\}
\]

We also need the definitions for a few other spaces which will make our computations convenient.

\[
\hat{A}_1 := \{(\tilde{f}, l_\tilde{p}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 : f(p) = 0, \nabla f|_p = 0, \nabla^2 f|_p(v, \cdot) \neq 0, \forall v \in l_\tilde{p}\}
\]

\[
\hat{D}_4 := \{(\tilde{f}, l_\tilde{p}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 : f(p) = 0, \nabla f|_p = 0, \nabla^2 f|_p = 0, \nabla^3 f|_p(v, v) \neq 0, \forall v \neq 0 \in l_\tilde{p}\}
\]

\[
\hat{X}_8 := \{(\tilde{f}, l_\tilde{p}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 : f(p) = 0, \nabla f|_p = 0, \nabla^2 f|_p = 0, \nabla^3 f|_p = 0, \nabla^4 f|_p(v, v, v) \neq 0 \forall v \neq 0 \in l_\tilde{p}\}
\]

\[
\hat{X}^\# := \{(\tilde{f}, l_\tilde{p}) \in \mathcal{D} \times \mathbb{P}T\mathbb{P}^2 : (\tilde{f}, l_\tilde{p}) \in \hat{X}_8^\#, \Psi_j(\tilde{f}, l_\tilde{p}) \neq 0, \text{where } \Psi_j \text{ is defined in (1.3.8) }\}
\]

1.4 Local structure of holomorphic sections

We give a necessary and sufficient criterion for a curve \( f^{-1}(0) \) to have a singularity of type \( \mathcal{X}_k \) at the point \( \tilde{p} \). Let \( \rho = \rho(x, y) \) be a holomorphic function defined on a neighborhood of the origin in \( \mathbb{C}^2 \) and \( i, j \) be non-negative integers. We define

\[
\rho_{ij} := \left. \frac{\partial^{i+j} \rho}{\partial x^i \partial y^j} \right|_{(x, y) = (0, 0)}.
\]

**Lemma 1.4.1.** Let \( \rho = \rho(x, y) \) be a holomorphic function defined on a neighborhood of the origin in \( \mathbb{C}^2 \) such that \( \rho_{00} = 0 \) and \( \nabla \rho|_{(0,0)} \neq 0 \). Then there exists a coordinate chart \( (u, v) \) centered at the origin so that \( \rho(u, v) = v^2 + u \).

**Proof:** Follows immediately by considering the Taylor expansion of \( \rho \). \( \square \)
**Corollary 1.4.2.** A curve $\rho^{-1}(0)$ has an $A_0$-node at the origin if and only if it satisfies the hypothesis of Lemma 1.4.1.

**Lemma 1.4.3.** Let $\rho = \rho(x, y)$ be a holomorphic function defined on a neighbourhood of the origin in $\mathbb{C}$ such that $\rho(0,0), \nabla \rho|_{(0,0)} = 0$ and $\nabla^2 \rho|_{(0,0)}$ is non-degenerate. Then there exists a coordinate chart $(u,v)$ centered at the origin so that $\rho(u,v) = v^2 + u^2$.

**Proof:** This is the Morse Lemma, which again follows by considering the Taylor expansion of $\rho$. □

**Corollary 1.4.4.** A curve $\rho^{-1}(0)$ has an $A_1$-node if and only if it satisfies the hypothesis of Lemma 1.4.3.

**Lemma 1.4.5.** Let $\rho = \rho(r,s)$ be a holomorphic function defined on a neighbourhood of the origin in $\mathbb{C}$ such that $\rho(0,0), \nabla \rho|_{(0,0)} = 0$ and there exists a non-zero vector $w = (w_1, w_2)$ such that at the origin $\nabla^2 f(w, \cdot) = 0$, i.e., the Hessian is degenerate. Let $x = w_1 r + w_2 s, y = -\bar{w}_2 r + \bar{w}_1 s$ and $\rho_{ij}$ be the partial derivatives with respect to the new variables $x$ and $y$. If $\rho_{02} \neq 0$, there exists a coordinate chart $(u,v)$ centered around the origin in $\mathbb{C}^2$ such that

$$\rho = \begin{cases} v^2, & \text{or} \\
v^2 + u^{k+1}, & \text{for some } k \geq 2. \end{cases} \quad (1.4.1)$$

**Remark 1.4.6.** In terms of the new coordinates we have $\rho_{00} = \rho_{10} = \rho_{01} = \rho_{20} = \rho_{11} = 0$ and $\rho_{02} \neq 0$. Here $\partial_x + 0 \partial_y = (1,0)$ is the distinguished direction along which the Hessian is degenerate.

**Proof:** Let the Taylor expansion of $\rho$ in the new coordinates be given by

$$\rho(x,y) = A_0(x) + A_1(x)y + A_2(x)y^2 + \ldots.$$ 

By our assumption on $\rho$, $A_2(0) \neq 0$. We claim that there exists a holomorphic function $B(x)$ such that after we make a change of coordinates $y = y_1 + B(x)$, the function $\rho$ is given by

$$\rho = \hat{A}_0(x) + \hat{A}_2(x)y_1^2 + \hat{A}_3(x)y_1^3 + \ldots$$

for some $\hat{A}_k(x)$ (i.e., $\hat{A}_1(x) \equiv 0$). To see this, we note that this is possible if $B(x)$ satisfies the identity

$$A_1(x) + 2A_2(x)B + 3A_3(x)B^2 + \ldots \equiv 0. \quad (1.4.2)$$

Since $A_2(0) \neq 0$, $B(x)$ exists by the Implicit Function Theorem \(^4\). Therefore, we can compute $B(x)$ as a power series using (1.4.2) and then compute $\hat{A}_0(x)$. Hence,

$$\rho = v^2 + \frac{A_0^2}{3!}x^3 + \frac{A_0^4}{4!}x^4 + \ldots, \quad \text{where} \quad v = \sqrt{(\hat{A}_2 + \hat{A}_3 y_1 + \ldots)y_1}, \quad (1.4.3)$$

satisfies (1.4.1). □

Following the above procedure we find $A_i^0$ for $i = 3, \ldots, 7$. In particular,

$$A_3^0 = \rho_{30}, \quad A_4^0 = \rho_{40} - \frac{3\rho_{21}^2}{\rho_{02}}, \quad A_5^0 = \rho_{50} - \frac{10\rho_{21}\rho_{31}}{\rho_{02}} + \frac{15\rho_{12}\rho_{21}^2}{\rho_{02}^2} \quad (1.4.4)$$

\(^4\)Moreover it is unique if we require $B(0) = 0.$
Corollary 1.4.7. Let the hypothesis be as in Lemma 1.4.5. The curve \( \rho^{-1}(0) \) has an \( A_k \)-node (for \( k \geq 2 \)) at the origin if and only if \( \rho_{02} \neq 0 \) and the directional derivatives \( A^p_1 \) obtained in (1.4.3) are zero for all \( i \leq k \) and \( A^p_{k+1} \neq 0 \). Furthermore, if \( \tau \) is any holomorphic function that does not vanish at the origin, then

\[
A^\rho_{k+1} = \tau_{00} A^\rho_{k+1} \quad \text{and} \quad (\tau \rho)_{02}^{k-3} A^\rho_{k+1} = \tau_{00}^{k-2} \rho_{02}^{k-3} A^\rho_{k+1}.
\]  
(1.4.5)

Finally, if \( A^\rho_i = 0 \) for \( i \leq k \) then the quantity \( A^\rho_{k+1} \) is invariant under

\[
x \rightarrow x + T_1(x,y), \quad y \rightarrow y + T_2(x,y) \quad \text{and} \quad x \rightarrow x + \tau\rho(x,v)
\]  
(1.4.6)

\[
y \rightarrow y + x, \quad x \rightarrow x
\]  
(1.4.7)

where \( T_1 \) and \( T_2 \) are holomorphic functions that vanish at the origin and whose derivative also vanish at the origin, i.e.,

\[
T_i(0,0) = 0, \quad \nabla T_i(0,0) = 0, \quad i = 1, 2.
\]

Proof: The first assertion follows immediately from (1.4.3). To prove (1.4.5), note that by (1.4.3)

\[
A^\rho_{k+1} = \frac{\partial^k \rho(x,v)}{\partial x^k} \bigg|_{(0,0)} \quad \Rightarrow \quad A^\tau\rho_{k+1} = \frac{\partial^{k+1} \tau \rho(x,v)}{\partial x^{k+1}} \bigg|_{(0,0)} = \tau_{00} A^\rho_{k+1}
\]

which follows from the fact that \( A^\rho_i = 0 \) for all \( i \leq k \). The second equation follows similarly by observing that \( (\tau \rho)_{02} = \tau_{00} \rho_{02} \). The proofs of (1.4.6) and (1.4.7) can be found in subsection 1.5.2.

Remark 1.4.8. The quantity \( \rho_{02}^{k-3} A^\rho_k \) is defined even when \( \rho_{02} = 0 \). These quantities induce sections \( \Psi_{\mathcal{P}A_k} \) of the line bundles \( \mathbb{L}_{\mathcal{P}A_k} \rightarrow \mathcal{D} \times \mathbb{P} \mathbb{T} \mathbb{P}^2 \) of subsection 1.3.3. The induced section is defined to be

\[
\{ \Psi_{\mathcal{P}A_k}(f, l_p) \} (f^\otimes (k - 2) \otimes p^\otimes d \otimes v^\otimes k \otimes w^\otimes (2k - 6)) := f_{02}^{k-3} A^f_k,
\]  
(1.4.8)

where \( A^f_k \) is the number we get by replacing \( \rho_{ij} \) by \( f_{ij} \) in \( A^\rho_k \) (\( f_{ij} \) is defined in (1.3.5)). Note that (1.4.5) and (1.4.6) imply that this section is well defined restricted to \( \Psi_{\mathcal{P}A_{k-1}}(0) \), i.e., it is independent of the trivialization and independent of the curve chosen. This is easily seen by unwinding definition 2.3.1. The details of this can be found in section 1.5. Finally, note that (1.4.7) implies that the section \( \Psi_{\mathcal{P}A_k} \) is well defined on the quotient space \( \mathbb{P} \mathbb{T} \mathbb{P}^2 / \mathbb{P}^1 \mathbb{P}^1 \), since the quantity in (1.4.8) is invariant under \( w \rightarrow w + v \).

Next we analyze singularities when the Hessian is identically zero.

Lemma 1.4.9. Let \( \rho = \rho(x, y) \) be a holomorphic function defined on a neighbourhood of the origin in \( \mathbb{C} \) such that \( \rho_{00}, \nabla \rho|_{(0,0)}, \nabla^2 \rho|_{(0,0)} = 0 \) and there does not exist a non-zero vector \( w = (w_1, w_2) \) such that at the origin \( \nabla^3 \rho(w, w, \cdot) = 0 \). Then, there exists a coordinate chart \( (u, v) \) centered at the origin so that \( \rho(u, v) = u^3 + v^3 \).

Proof: The Taylor expansion of \( \rho \) is given by

\[
\rho = \frac{\rho_{00}}{6} x^3 + \frac{\rho_{21}}{2} x^2 y + \frac{\rho_{12}}{2} xy^2 + \frac{\rho_{03}}{6} y^3 + \ldots
\]

We make an observation from multi-linear algebra that if \( \nabla^3 \rho \) is non-degenerate, then the cubic term in the Taylor expansion has no repeated factors (this analogous to the similar fact that if \( \nabla^2 \rho \) is
non-degenerate, then the quadratic term in the Taylor expansion is not a perfect square). Hence, we can make a linear change of coordinates so that $\rho$ is given by

$$\rho = x^3 + y^3 + \eta x^2 y_1 + x^3 h_1 + y^3 h_2$$

where $h_1$ and $h_2$ are holomorphic functions of $x_1$ and $y_1$ vanishing at the origin and $\eta$ is some number. We now make a change of coordinate $x_1 = x_2 + Ay_1^2$ to get rid of the coefficient of $x^2 y_1^2$. Equating coefficients we get $3A + \eta = 0$. Hence,

$$\rho = x^3 + y_1^3 + x^3 h_3 + y_1^3 h_4,$$

where $h_3$ and $h_4$ are holomorphic functions of $x_2$ and $y_1$ that vanish at the origin. This is equivalent to $u^3 + v^3$ after a change of coordinates. \qed

**Corollary 1.4.10.** A curve $\rho^{-1}(0)$ has a D4-node if and only if it satisfies the hypothesis of Lemma 1.4.9.

**Lemma 1.4.11.** Let $\rho = \rho(r, s)$ be a holomorphic function defined on a neighbourhood of the origin in $\mathbb{C}$ such that $\rho_{00}, \nabla \rho_{0(0,0)}, \nabla^2 \rho_{(0,0)} = 0$ and there exists a non-zero vector $w = (w_1, w_2)$ such that at the origin $\nabla^2 \rho(w, w, \cdot) = 0$. Let $x = w_1 r + w_2 s$, $y = -\overline{w}_2 r + \overline{w}_1 s$ and $\rho_{ij}$ be the partial derivatives with respect to the new variables $x$ and $y$. If $\rho_{12} \neq 0$, there exists a coordinate chart $(u, v)$ centered around the origin in $\mathbb{C}^2$ such that

$$\rho(u, v) \equiv \left\{ \begin{array}{ll} v^2 u & \text{or} \\ v^2 u + u^{k-1} & \text{for some } k \geq 5. \end{array} \right.$$

Note that in terms of the new coordinates we have

$$\rho_{00} = \rho_{10} = \rho_{01} = \rho_{20} = \rho_{11} = \rho_{02} = \rho_{30} = \rho_{21} = 0, \rho_{12} \neq 0.$$

**Proof:** In terms of the new coordinates $x$ and $y$, the Taylor expansion of $\rho$ is given by

$$\rho = \frac{\rho_{12}}{2} xy^2 + \frac{\rho_{03}}{6} y^3 + \frac{\rho_{40}}{24} x^4 + \ldots$$

We claim that there exists a holomorphic function $G(y)$, such that after making a change of coordinate $x = x_1 + G(y)$, the function is given by $\rho = x_1 g(x_1, y)$, i.e., we can kill off all powers of $y$. Assuming such a $G(y)$ exists, we can now apply the argument as in Lemma 1.4.5. Let

$$g(x_1, y) = A_0(x_1) + A_1(x_1)y + A_2(x_1)y^2 + \ldots.$$ 

We can make a change of coordinate $y = y_1 + B(x_1)$ so that

$$g = \hat{A}_0(x_1) + A_2(x_1)y_1^2 + \ldots$$

i.e., $\hat{A}_1(x_1) \equiv 0$. This is possible since $A_2(0) \neq 0^5$. That gives us

$$\rho = x_1(\hat{A}_0(x_1) + \hat{A}_2(x_1)y_1^2 + \hat{A}_3(x_1)y_1^3 + \ldots)$$

---

\[ ^5 \text{Notice that } A_2(0) = \frac{\rho_{40}}{24} \text{ is non-zero by hypothesis. } \]
If we set
\[ \hat{A}_0(x_1) = \frac{D^\rho}{4!} x_1^3 + \frac{D^\rho}{5!} x_1^4 + \ldots \]
then \( \rho \) is given by
\[ \rho = x_1 \left( \frac{D^\rho_2}{4!} x_1^3 + \frac{D^\rho_2}{5!} x_1^4 + \ldots \right) \]
If \( \hat{A}_0(x_1) \equiv 0 \) then \( \rho = x_1 y_2^2 \) is of the intended form. Otherwise let \( k \) be the smallest integer such that \( D^\rho_{k+1} \neq 0 \). Let
\[ x_2 = \sqrt{k^{-1} \frac{D^\rho_{k+1}}{(k-1)!} x_1^{k-1} + \frac{D^\rho_{k+2}}{k!} x_1^k + \ldots} \]  
with \( x_1 = Cx_2 + O(x_2^2) \) and \( C = ((k-1)!/D^\rho_{k+1})^{k-1} \). In these new coordinates, \( \rho \) is given by
\[ \rho = (Cx_2 + x_2^2 h) y_2^2 + x_2^{k-1} \]
for some holomorphic function \( h(x_2, y_2) \). Now define \( y_3 = y_2 \sqrt{C + x_2 h} \). Therefore,
\[ \rho = y_3^2 x_2 + x_2^{k-1} \]  
(1.4.11)
to get the intended form as in (1.4.11).

It remains to show that \( G(y) \) exists. By our assumption on \( \rho \), we know that \( \rho_{30} = \rho_{21} = 0 \) and \( \rho_{12} \neq 0 \). Hence, the Taylor expansion of \( \rho \) is given by
\[ \rho(x, y) = P_{12}(x, y) xy^2 + P_{03}(y) y^3 + P_{40}(x, y) x^4 + P_{31}(x, y) x^3 y \]  
(1.4.12)
for some holomorphic functions \( P_{ij} \) with \( P_{12}(0, 0) \neq 0 \). Recall that we want \( x = x_1 + G(y) \) so that \( \rho = x_1 g(x_1, y) \), i.e., the coefficients of \( y^n \) are killed for all \( n \). This is equivalent to saying that we want to find a \( G \) such that \( \rho(G(y), y) = 0^6 \). Plugging in \( x = x_1 + yH(y) \) in (1.4.12) we get that
\[ 0 = \rho(yH(y), y) = P_{12}(yH(y), y)y^3 H(y) + P_{03}(y)y^3 + P_{40}(yH(y), y)y^4 H(y)^4 + P_{31}(yH(y), y)y^3 H(y). \]
This implies that
\[ P_{12}(yH(y), y) H(y) + P_{03}(y) + P_{40}(yH(y), y)yH(y)^4 + P_{31}(yH(y), y)H(y) = 0 \]
By the implicit function theorem, \( H(y) \) exists since \( P_{12}(0, 0) \neq 0 \), whence \( G(y) = yH(y) \) exists. \( \square \)

In practice we first find \( H(y) \) as a power series and then find \( A(x_1) \) as a power series. That ultimately gives us \( D^\rho_i \). Following the above procedure, we prove (1.3.10).

**Corollary 1.4.12.** Let the hypothesis be as in Lemma 1.4.11. Then the curve \( \rho^{-1}(0) \) has a \( D_k \)-node if and only if the directional derivatives \( D^\rho_i \) obtained in (1.4.9) are zero for all \( i \leq k \) and \( D^\rho_k \neq 0 \). Furthermore, if \( \tau \) is a holomorphic function that does not vanish at the origin, then
\[ D^\tau_{k+1} = \tau_{00} D^\rho_{k+1} \quad \text{and} \quad (\tau \rho)^{k-6} D^\tau_{k+1} = \tau_{00}^k \rho_{12}^{k-6} D^\rho_{k+1}. \]  
(1.4.13)

\(^6\)Note that \( \rho(G(y), y) \) gives us the part of the Taylor expansion of \( \rho(x_1 + G(y), y) \) that only depends on \( y \). We desire that part to be zero, which is equivalent to killing off all the \( y^n \) terms in the expansion of \( \rho(x_1 + G(y), y) \).
Finally, if $D_i^\rho = 0$ for $i \leq k$ then the quantity $D_{k+1}^\rho$ is invariant under

$$x \rightarrow x + T_1(x,y), \quad y \rightarrow y + T_2(x,y)$$

\[ (1.4.14) \]

$$x \rightarrow x, \quad y \rightarrow y + x, \quad \text{where} \ T_1 \text{ and } T_2 \text{ are holomorphic functions that vanish at the origin and whose derivative also vanish at the origin, i.e.,}$$

\[ T_i(0,0) = 0, \quad \nabla T_i(0,0) = 0, \quad \text{where } i = 1,2. \]

**Proof:** The first assertion follows immediately from (1.4.11). To prove the second assertion, note that by (1.4.9)

\[ D_{k+1}^\rho = \frac{\partial^{k+1}\rho(x_1,y_2)}{\partial x_1^{k+1}} \bigg|_{(0,0)} \implies D_{k+1}^\tau = \frac{\partial^{k+1}\tau(x_1,y_2)}{\partial x_1^{k+1}} \bigg|_{(0,0)} = \tau_0 D_{k+1}^\rho. \]

which follows from the product rule and the fact that $D_i^\rho = 0$ for all $i \leq k$. The second equation follows similarly by observing that $(\tau \rho)_{12} = \tau_0 \rho_{12}$. The proofs of (1.4.14) and (1.4.15) can be found in 1.5.2.

**Remark 1.4.13.** Similar to remark 1.4.8, the quantities $\rho_{12}^\epsilon D_k^\rho$ induce sections of the line bundle $\mathbb{L}_{PD_k} \rightarrow \mathcal{D} \times \mathcal{P} T^{\rho_2^2}$ given by

\[ \{\Psi_{PD_k}(f, l_\rho)\} \left( f^{\rho d(1+\epsilon_k)} \otimes \nu^{\rho \rho d(1+\epsilon_k)} \otimes v^\rho \otimes (2\varepsilon_\rho) \right) = f_{12}^\rho \Psi_{PD_k} \]

\[ (1.4.16) \]

where $\varepsilon_\rho$ is the order of the pole of $D_i^\rho$ at $\rho_{12} = 0$. Equations (1.4.13) and (1.4.14) imply that this section is well defined restricted to $\Psi_{PD_k}^{-1}(0)$. Equation (1.4.15) implies that the section $\Psi_{PD_k}$ is well defined on the quotient space.

Next we analyze the singularities $\mathcal{E}_6$ and $\mathcal{E}_7$.

**Lemma 1.4.14.** Let $\rho = \rho(r,s)$ be a holomorphic function defined on a neighbourhood of the origin in $\mathbb{C}$ such that $\rho_{10} = \nabla \rho |_{(0,0)} = \nabla^2 \rho |_{(0,0)} = 0$ and there exists a non-zero vector $w = (w_1, w_2)$ such that at the origin $\nabla^2 \rho (w,w, \cdot) = 0$. Let $x = w_1 r + w_2 s$, $y = -w_2 r + w_1 s$ and $\rho_{ij}$ be partial derivatives with respect to the new coordinates, $x$ and $y$. If $\rho_{12} = 0$ and $\rho_{30} \neq 0, \rho_{40} \neq 0$, there exists a coordinate chart $(u,v)$ centered at the origin so that

\[ \rho(u,v) = v^3 + u^4. \]

\[ (1.4.17) \]

Note that in terms of the new coordinates $x$ and $y$, we get that

\[ \rho_{00} = \rho_{10} = \rho_{01} = \rho_{20} = \rho_{11} = \rho_{02} = \rho_{30} = \rho_{21} = \rho_{12} = 0, \rho_{30} \neq 0, \rho_{40} \neq 0. \]

**Proof:** The Taylor expansion of $\rho$ is given by,

\[ \rho(x,y) = P_{03}(x,y)y^3 + P_{40}(x,y)x^4 + \kappa_1 x^3 y + \kappa_2 x^2 y^2 + \kappa_3 x^3 y^2 \]

\[ (1.4.18) \]

in terms of the new variables $x$ and $y$, for some constants $\kappa_1, \kappa_2, \kappa_3$ and holomorphic functions $P_{03}$ and $P_{40}$ on a neighborhood of the origin in $\mathbb{C}^2$ such that $P_{03}(0,0), P_{40}(0,0) \neq 0$. We claim that there exists constants $\eta_1, \eta_2$ and $\eta_3$, such that if we make the substitution

\[ x = \hat{x} + \eta_1 \hat{y}, \quad y = \hat{y} + \eta_2 \hat{x}^2 + \eta_3 \hat{x}^3 \]
then \( \rho \) is given by

\[
\rho = \hat{P}_{03}(\hat{x}, \hat{y})\hat{y}^3 + \hat{P}_{40}(\hat{x}, \hat{y})\hat{x}^4
\]  \quad (1.4.19)

such that \( \hat{P}_{03}(0,0), \hat{P}_{40}(0,0) \neq 0 \). To see this, we note that this is possible if the following equations are satisfied:

\[
\frac{\rho_{40}}{6} \eta_1 + \kappa_1 = 0, \quad \frac{\rho_{03}}{2} \eta_2 + \frac{\rho_{40}}{4} \eta_1^2 + 3\kappa_1 \eta_1 + \kappa_2 = 0,
\]

\[
\frac{\rho_{03}}{2} \eta_3 + \frac{\rho_{13}}{2} \eta_2 + \frac{\rho_{50}}{12} \eta_1^2 + \frac{\rho_{41}}{6} \eta_1 + 3\kappa_1 \eta_1 \eta_2 + 4\kappa_2 \eta_1 \eta_2 + \kappa_3 = 0.
\]

Solutions to \( \eta_1, \eta_2, \eta_3 \) exist, since \( \rho_{40} \neq 0 \) and \( \rho_{03} \neq 0 \). It is easy to see that (1.4.19) is equivalent to (1.4.17) after a change of coordinates, since \( \hat{P}_{03}(0,0), \hat{P}_{40}(0,0) \neq 0 \).

**Corollary 1.4.15.** A curve \( \rho^{-1}(0) \) has an \( \mathcal{E}_0 \)-node if and only if it satisfies the hypothesis of Lemma 1.4.14.

**Lemma 1.4.16.** Let \( \rho = \rho(r,s) \) be a holomorphic function defined on a neighbourhood of the origin in \( \mathbb{C} \) such that \( \rho_{00}, \nabla \rho|_{(0,0)}, \nabla^2 \rho|_{(0,0)} = 0 \) and there exists a non-zero vector \( w = (w_1, w_2) \) such that at the origin \( \nabla^2 \rho(w,w,\cdot) = 0 \). Let \( x = w_1 r + w_2 s, \ y = -w_2 r + w_1 s \). Let \( \rho_{ij} \) be the partial derivatives with respect to the new variables \( x \) and \( y \). If \( \rho_{12} = \rho_{40} = 0 \) and \( \rho_{03} \neq 0, \rho_{31} \neq 0 \), then there exists a coordinate chart \( (u,v) \) centered at the origin so that

\[
\rho(u,v) = v^3 + u^3 v.
\]  \quad (1.4.20)

In terms of the new coordinates \( x \) and \( y \), we get that

\[
\rho_{00} = \rho_{10} = \rho_{01} = \rho_{20} = \rho_{11} = \rho_{02} = \rho_{30} = \rho_{21} = \rho_{12} = \rho_{40} = 0, \rho_{03} \neq 0, \rho_{31} \neq 0.
\]

**Proof:** The Taylor expansion of \( \rho \) is given by,

\[
\rho(x,y) = P_{03}(x,y)\eta_3 + P_{31}(x,y)\eta_3 y + \kappa x^2 y^2 + P_{50}(x)x^5
\]  \quad (1.4.21)

in terms of the new coordinates \( x \) and \( y \), for constant \( \kappa \) and holomorphic functions \( P_{03}, \ P_{31}, \) and \( P_{50} \) such that \( \rho_{03}(0,0), \rho_{31}(0,0) \neq 0 \). We claim that there exists a holomorphic function \( B(\hat{x}) \) and constant \( \eta_1 \) such that if we make the substitution

\[
\hat{x} = \hat{x} + \eta_1 \hat{y}, \quad \hat{y} = \hat{y} + B(\hat{x})\hat{x}^2
\]

then \( \rho \) is given by

\[
\rho = \hat{P}_{03}(\hat{x}, \hat{y})\hat{y}^3 + \hat{P}_{31}(\hat{x}, \hat{y})\hat{x}^3 \hat{y}
\]  \quad (1.4.22)

To see this, note that this is possible if the following equations are satisfied:

\[
P_{31}(\hat{x}, B(\hat{x})\hat{x}^2)B(\hat{x}) + P_{03}(\hat{x}, B(\hat{x})\hat{x}^2)\hat{x}B(\hat{x})^3 + k\hat{x}B(\hat{x})^2 + P_{50}(\hat{x}) = 0,
\]

\[
\frac{\rho_{31}}{2} \eta_1 + \frac{\rho_{03}}{2} B(0) + \kappa = 0.
\]

A solution to \( B(\hat{x}) \) exists since \( \rho_{31} \neq 0 \). We see that

\[
B(0) = -\frac{P_{50}(0)}{P_{31}(0,0)} = -\frac{\rho_{50}}{20\rho_{31}}
\]

which implies the existence of \( \eta_1 \). It is easy to see that (1.4.22) is equivalent to (1.4.20) after a change of coordinates, since \( \hat{P}_{03}(0,0), \hat{P}_{31}(0,0) \neq 0 \). \( \square \)
Corollary 1.4.17. A curve $\rho^{-1}(0)$ has an $E_7$-node if and only if it satisfies the hypothesis of Lemma 1.4.16.

Let us now summarize certain facts about sections of vector bundles, involving the vertical derivative.

Lemma 1.4.18. Let $L \rightarrow M$ a complex line bundle over a two dimensional complex manifold $M$, $s: M \rightarrow L$ a holomorphic section and $q \in M$ a point in $M$. Let $v, w \in T_q M$ be two tangent vectors at the point $q$. Then the following are true:

1. If $s(q) = 0$ and $\nabla^3 s|_q = 0$ for all $i < k$ then $\nabla^k s|_q$ is well defined. Furthermore, for any tangent vectors $v, w \in T_q M$ and non-negative integers $i$ and $j$ such that $i + j = k$, the quantity

$$\tilde{s}_{ij} := \nabla^{i+j}s|_q(v, \ldots, v, w, \ldots w)$$

(1.4.23)

is also well defined.

2. If $s(q) = 0$, $\nabla s|_q = 0$ and $\nabla^2 s|_q(v, \cdot) = 0$ then $\nabla^3 s|_q(v, v, v)$ is also well defined.

3. Let $A^s_k$ be the corresponding sections induced from the quantities $A^p_k$ obtained in (1.4.3) by replacing $\rho_j$ with $\tilde{s}_{ij}$. If $s(q) = 0$, $\nabla s|_q = 0$, $\nabla^2 s|_q = 0$ and $\tilde{s}_{02}^{i-k} A^s_k = 0$ for all $i < k$, then $\tilde{s}_{02}^{i-k} A^s_k$ is well defined. Furthermore, the quantity $\tilde{s}_{02}^{i-k} A^s_k$ is invariant under $w \rightarrow w + v$.

4. If $s(q) = 0$, $\nabla s|_q = 0$, $\nabla^2 s|_q = 0$ and $\nabla^3 s|_q(v, v, v) = 0$, then $\nabla^4 s|_q(v, v, v, v)$ is well defined.

5. Let $D^s_k$ be the corresponding sections induced from the quantities $D^p_k$ obtained in (1.4.9) by replacing $\rho_j$ with $\tilde{s}_{ij}$. If $s(q) = 0$, $\nabla s|_q = 0$, $\nabla^2 s|_q = 0$, $\nabla^3 s|_q(v, v, v) = 0$ and $\tilde{s}_{12}^{i-k} D^s_k = 0$ for all $i < k$, then $\tilde{s}_{12}^{i-k} D^s_k$ is well defined. Furthermore, the quantity $\tilde{s}_{12}^{i-k} D^s_k$ is invariant under $w \rightarrow w + v$. Here as before, $\epsilon_k$ is the order of the pole of $D^p_k$ at $\rho_{12} = 0$.

6. If $s(q) = 0$, $\nabla s|_q = 0$, $\nabla^2 s|_q = 0$ and $\nabla^3 s|_q(v, v, v) = 0$, then $\nabla^4 s|_q(v, v, v, v)$ is well defined. Furthermore, the quantity $\nabla^4 s|_q(v, v, v)$ is invariant under $w \rightarrow w + v$.

7. If $s(q) = 0$, $\nabla s|_q = 0$, $\nabla^2 s|_q = 0$, $\nabla^3 s|_q(v, v, v) = 0$, and $\nabla^3 s|_q(v, w, w) = 0$, then $\nabla^4 s|_q(v, v, v)$ is well defined.

8. If $s(q) = 0$, $\nabla s|_q = 0$, $\nabla^2 s|_q = 0$, $\nabla^3 s|_q(v, v, v) = 0$, $\nabla^3 s|_q(v, w, w) = 0$ and $\nabla^4 s|_q(v, v, v) = 0$, then $\nabla^4 s|_q(v, v, v, v)$ is well defined. Furthermore, the quantity $\nabla^3 s|_q(v, v, v)$ is invariant under $w \rightarrow w + v$.

**Proof:** See section 1.5. These facts follow in a straightforward way by unwinding definition 2.3.1. As explained in remarks 1.4.8 and 1.4.13, Corollary 1.4.7 and 1.4.12 imply Lemma 1.4.18, statement 3 and 5, respectively.

**Remark 1.4.19.** Let us mention a pedantic point about our notation. The $\cdot$ introduced in the notation of (1.4.23) might seem strange to the reader. We have done that to be consistent with (1.3.5). According to our notation, if $f: \mathbb{P}^2 \rightarrow \gamma^{d}_{p^d}$ is a section and $\tilde{p} \in \mathbb{P}^2$ then

$$f_{ij} := \nabla^{i+j} f|_{\tilde{p}}(v, \ldots, v, w, \ldots w) \in \gamma^{d}_{p^d}|_{\tilde{p}}$$

and

$$f_{ij} := \nabla^{i+j} f|_{\tilde{p}}(v, \ldots, v, w, \ldots w) \in \mathbb{C}.$$
Since we encounter the second quantity more in our computations, we have denoted that as $f_{ij}$. Notice that if $\hat{f}_{ij}$ is well defined, then so is $f_{ij}$.

1.5 Proof of Lemma 1.4.18: Sections are well defined

We will now prove Lemma 1.4.18. The main content of that Lemma is that all the sections we have obtained are indeed well defined (restricted to certain subspaces). In order to prove that, we show that it is independent of the trivialization and independent of the curve. Furthermore, the sections are also well defined on the quotient space.

1.5.1 Sections are independent of the trivialization and the curve

First, we will recall the general setup.

Let $\pi : V \to M$ be a holomorphic vector bundle of rank $k$ and $s : M \to V$ a holomorphic section. Suppose $h_\alpha : V |_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{C}^k$ is a holomorphic trivialization of $V$ and $\pi_1, \pi_2 : \mathcal{U}_\alpha \times \mathbb{C}^k \to \mathcal{U}_\alpha, \mathbb{C}^k$ the projection maps. Let $s_\alpha = \pi_2 \circ h_\alpha \circ s$. Thus, the diagram

\[
\begin{array}{ccc}
\pi^{-1}(\mathcal{U}_\alpha) & \xrightarrow{h_\alpha} & \mathcal{U}_\alpha \times \mathbb{C}^k \\
\downarrow s & \searrow \pi & \downarrow \pi_2 \\
\mathcal{U}_\alpha & \xrightarrow{s_\alpha} & \mathbb{C}^k
\end{array}
\]

commutes. For $p \in \mathcal{U}_\alpha$, we define the vertical derivative of $s$ to be the $\mathbb{C}$-linear map

$\nabla s|_p : T_p M \to V_p, \quad \nabla s|_p = (\pi_2 \circ h_\alpha)|^{-1} \circ ds_\alpha|_p,$

where $V_p = \pi^{-1}(p)$, the fiber at $p$. More generally, if $v, w \in T_p M$ are tangent vectors such that there exists a family of complex curves $\gamma : B_\epsilon \times B_\epsilon \to M$ such that

$\gamma(0,0) = p, \quad \frac{\partial \gamma(x,y)}{\partial x}\bigg|_{(0,0)} = v, \quad \frac{\partial \gamma(x,y)}{\partial y}\bigg|_{(0,0)} = w$

then

$\hat{s}_{ij} := \nabla^{i+j}s|_p(v, \ldots, v, w, \ldots, w) := (\pi_2 \circ h_\alpha)|^{-1} \circ \left[\frac{\partial^{i+j}s_\alpha(\gamma(x,y))}{\partial x^i \partial y^j}\right]|_{(0,0)}.$

(1.5.2)

where $B_\epsilon$ is an open ball around the origin in $\mathbb{C}$ of radius $\epsilon$. Combined with the fact that $\nabla s|_p$ is $\mathbb{C}$-linear, it determines the map $\nabla s|_p$ completely.

In general the quantity $\hat{s}_{ij}$ is not well defined, since it depends on the curve and it depends on the trivialization. First let us explain what we need to prove, to show that $\hat{s}_{ij}$ is well defined on some subspace. The basic principal is the same in all the cases. First let us keep the curve same, but change the trivialization. Since we only require the special case where $V$ is a line bundle, we work that out. The general principal is same for vector bundles. Let $s_\beta : \mathcal{U}_\beta \to \mathbb{C}$ be the section seen in a different coordinate chart and trivialization. Define a map

$g_{\alpha\beta} : (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \to \text{GL}(1, \mathbb{C}) \quad \text{by} \quad g_{\alpha\beta}(p) = (\pi_2 \circ h_\alpha)|^{-1} \circ (\pi_2 \circ h_\beta)|^{-1}.$
It is easy to see that
\[ s_\beta(p) = g_\beta(p) s_\alpha(p). \]

In order to show that the quantity \( \dot{s}_{ij} \) is well defined, we need to show that
\[
(\pi_2 \circ h_\beta) \big|_{V^1_p} \frac{\partial^{i+j}s_\beta(\gamma(x, y))}{\partial x^i y^j} \bigg|_{(0,0)} = (\pi_2 \circ h_\alpha) \big|_{V^1_p} \frac{\partial^{i+j}s_\alpha(\gamma(x, y))}{\partial x^i y^j} \bigg|_{(0,0)} \tag{1.5.3}
\]

To make this a little bit easier to understand, let us denote the function \( s_\alpha(\gamma(x, y)) \) as \( \rho(x, y) \). Let us also denote \( g_\beta(\gamma(x, y)) \) as \( \tau(x, y) \). Hence, what we need to show is that
\[
(\pi_2 \circ h_\beta) \big|_{V^1_p} \frac{\partial^{i+j}\tau(x, y)\rho(x, y)}{\partial x^i y^j} \bigg|_{(0,0)} = (\pi_2 \circ h_\alpha) \big|_{V^1_p} \frac{\partial^{i+j}\rho(x, y)}{\partial x^i y^j} \bigg|_{(0,0)} \tag{1.5.4}
\]

More generally, if \( \Phi(\dot{s}_{ij}) \) is a homogeneous degree \( n \) polynomial in the \( \dot{s}_{ij} \), then to show that \( \Phi(\dot{s}_{ij}) \) is well defined, we need to show that
\[
\Phi((\tau\rho)_{ij}) = \tau(0,0)^n \Phi(\rho_{ij}) \tag{1.5.5}
\]
where \( (\tau\rho)_{ij} \) is the partial derivative of the function \( \tau(x, y)\rho(x, y) \) at the origin.

Next, suppose the trivialization is the same, but we change the curve. Then, we need to show that, if \( \tilde{\gamma} \) is another curve, such that
\[
\tilde{\gamma}(0,0) = p, \quad \frac{\partial\tilde{\gamma}(x, y)}{\partial x} \bigg|_{(0,0)} = v, \quad \frac{\partial\tilde{\gamma}(x, y)}{\partial y} \bigg|_{(0,0)} = w
\]
then
\[
(\pi_2 \circ h_\beta) \big|_{V^1_p} \frac{\partial^{i+j}s_\beta(\gamma(x, y))}{\partial x^i y^j} \bigg|_{(0,0)} = (\pi_2 \circ h_\alpha) \big|_{V^1_p} \frac{\partial^{i+j}s_\alpha(\gamma(x, y))}{\partial x^i y^j} \bigg|_{(0,0)} \tag{1.5.6}
\]

Again for simplicity let us denote the two functions \( s_\alpha(\gamma(x, y)) \) and \( s_\alpha(\tilde{\gamma}(x, y)) \) as \( \rho(x, y) \) and \( \tilde{\rho}(x, y) \).

Let
\[ T_1, T_2 : B_x \times B_\varepsilon \rightarrow \mathbb{C} \]
be holomorphic maps that vanish the origin and whose first derivatives also vanish at the origin, i.e.
\[
T_1(0,0) = 0, \quad \nabla T_1(0,0) = 0 \\
T_2(0,0) = 0, \quad \nabla T_2(0,0) = 0.
\]

(In this case \( \nabla \) is the usual calculus \( \nabla \) and \( B_\varepsilon \) denotes an open epsilon ball around the origin in \( \mathbb{C} \).)

Then we have that
\[ \tilde{\rho}(x, y) = \rho(x + T_1(x, y), y + T_2(x, y)) \]
for some such \( T_1 \) and \( T_2 \). Hence, to show that an expression involving \( \dot{s}_{ij} \) is independent of the curve is equivalent to showing the the corresponding expression involving \( \rho_{ij} \) is invariant under
\[
x \mapsto x + T_1(x, y), \quad y \mapsto y + T_2(x, y).
\]
Now, we are ready to prove Lemma 1.4.18.

In the subsequent discussions, it is understood that \( \tau(x, y) \) is a non zero holomorphic function in a neighbourhood of the origin and \( T_1(x, y) \) and \( T_2(x, y) \) are holomorphic functions that vanish at the origin and whose derivatives also vanish at the origin.

**Proof of Lemma 1.4.18, statement 1:** Let \( \rho(x, y) \) be a holomorphic function defined on a neighborhood of the origin such that all the partial derivatives of order less than \( k \) vanish at the origin. Let \( \tau(x, y) \) be a non zero holomorphic function defined on a neighborhood of the origin. The Taylor expansion of \( \rho(x, y) \) is given by

\[
\rho(x, y) = \frac{\rho_{00}}{k!} x^k + \frac{\rho_{k-1,1}}{(k-1)!} x^{k-1} y + \frac{\rho_{k-2,2}}{(k-2)!} x^{k-2} y^2 + \cdots + \frac{\rho_{0k}}{k!} y^k + \cdots
\]

It is now easy to see by looking at the Taylor expansion, that

\[
(\tau \rho)_{ij} = \tau_{00} \rho_{ij} \quad \forall \ i + j = k
\]

where \( \tau(x, y) \) is a non zero holomorphic function in a neighbourhood of the origin and \( T_1(x, y) \) and \( T_2(x, y) \) are holomorphic functions that vanish at the origin and whose derivatives also vanish at the origin. As per the previous discussion, this proves the Lemma 1.4.18, statement 1.

**Proof of Lemma 1.4.18, statement 2:** Let \( \rho(x, y) \) be a holomorphic function defined on a neighborhood of the origin such that it has the following Taylor expansion:

\[
\rho(x, y) = \frac{\rho_{02}}{2} y^2 + \frac{\rho_{30}}{6} x^3 + \cdots
\]

It is easy to see that

\[
(\tau \rho)_{30} = \tau_{00} \rho_{30}
\]

\[
\rho_{30} = \left. \frac{\partial^3 \rho(x + T_1(x, y), y + T_2(x, y))}{\partial^3 x} \right|_{(0,0)}.
\]

As per the previous discussion, this proves the Lemma.

**Proof of Lemma 1.4.18, statement 3:** Follows from Corollary 1.4.7. The fact that the quantity \( \mathcal{A}_k^\rho \) is invariant under \( x \rightarrow x + T_1(x, y) \) and \( y \rightarrow y + x + T_2(x, y) \) is shown in subsection 1.5.2. In particular, this implies that the induced section is invariant under \( w \rightarrow w + v \), i.e. it is well defined on the quotient space, in addition to being independent of the trivialization and choice of curve.

**Proof of Lemma 1.4.18, statement 4:** As before, let \( \rho(x, y) \) be a holomorphic function defined on a neighborhood of the origin such that it has the following Taylor expansion:

\[
\rho(x, y) = \frac{\rho_{12}}{2} xy^2 + \frac{\rho_{03}}{6} y^3 + \frac{\rho_{40}}{24} x^4 + \cdots
\]

It is easy to see that

\[
(\tau \rho)_{40} = \tau_{00} \rho_{40}
\]

\[
\rho_{40} = \left. \frac{\partial^4 \rho(x + T_1(x, y), y + T_2(x, y))}{\partial^4 x} \right|_{(0,0)}.
\]
As per the previous discussion, this proves the Lemma.

**Proof of Lemma 1.4.18, statement 5:** Follows from Corollary 1.4.12. The fact that the quantity $D_k^0$ is invariant under $x \rightarrow x + T_1(x, y)$ and $y \rightarrow y + x + T_2(x, y)$ is shown in subsection 1.5.2. In particular, this implies that the induced section is invariant under $w \rightarrow w + v$, i.e. it is well defined on the quotient space, in addition to being independent of the trivialization and choice of curve.

**Proof of Lemma 1.4.18, statement 6:** As before, let $\rho(x, y)$ be a holomorphic function defined on a neighborhood of the origin such that it has the following Taylor expansion:

$$
\rho(x, y) = \frac{\rho_{12}}{2} xy^2 + \frac{\rho_{03}}{6} y^3 + \frac{\rho_{40}}{24} x^4 + \ldots
$$

It is easy to see that

$$(\tau \rho)_{12} = \tau_{00} \rho_{12}
\rho_{12} = \left. \frac{\partial^3 \rho(x + T_1(x, y), y + x + T_2(x, y))}{\partial x \partial y^2} \right|_{(0,0)} .
$$

(1.5.10)

In particular, this implies that the induced section is invariant under $w \rightarrow w + v$ and hence is well defined on the quotient space, in addition to being independent of the trivialization and choice of curve.

**Proof of Lemma 1.4.18, statement 7:** As before, let $\rho(x, y)$ be a holomorphic function defined on a neighborhood of the origin such that it has the following Taylor expansion:

$$
\rho(x, y) = \frac{\rho_{03}}{6} y^3 + \frac{\rho_{40}}{24} x^4 + \ldots
$$

It is easy to see that

$$(\tau \rho)_{40} = \tau_{00} \rho_{40}
\rho_{40} = \left. \frac{\partial^4 \rho(x + T_1(x, y), y + x + T_2(x, y))}{\partial x^4} \right|_{(0,0)} .
$$

(1.5.11)

As per the previous discussion, this proves the Lemma.

**Proof of Lemma 1.4.18, statement 8:** As before, let $\rho(x, y)$ be a holomorphic function defined on a neighborhood of the origin such that it has the following Taylor expansion:

$$
\rho(x, y) = \frac{\rho_{03}}{6} y^3 + \frac{\rho_{31}}{6} x^3 y + \frac{\rho_{22}}{4} x^2 y^2 + \frac{\rho_{13}}{6} x y^3 + \frac{\rho_{04}}{24} y^4 + \frac{\rho_{50}}{120} x^5 + \ldots
$$

It is easy to see that

$$(\tau \rho)_{31} = \tau_{00} \rho_{31}
\rho_{31} = \left. \frac{\partial^4 \rho(x + T_1(x, y), y + x + T_2(x, y))}{\partial x^3 \partial y} \right|_{(0,0)} .
$$

(1.5.12)

In particular, this implies that the induced section is invariant under $w \rightarrow w + v$ and hence is well defined on the quotient space. As per the previous discussion, this proves the Lemma.
1.5.2 Proof of Corollary 1.4.7 and 1.4.12

We will now give the proof of Corollary 1.4.7 and 1.4.12. In particular, this will show that the sections we have obtained are well defined on the quotient space (in addition to being well defined under a change of coordinates/curves and change of trivializations).

In all these discussions $T_1$, $T_2$, $\tilde{T}_1$ and $\tilde{T}_2$, will denote holomorphic functions of two variables, that vanish at the origin and whose derivatives also vanish at the origin.

Detailed proof of Corollary 1.4.7

Recall that Corollary 1.4.7 claims that the induced sections $\psi_{\mathcal{P}A_k}$ are well defined. Let $\rho : \mathbb{C}^2 \rightarrow \mathbb{C}$ be a holomorphic function with the following properties:

$$\rho_{00}, \rho_{10}, \rho_{01} = 0, \quad \rho_{02} \neq 0.$$  

Initially our coordinates are $x$ and $y$. The new coordinates are $x$ and $z = y - x - \tilde{T}_2(x, y)$, where $\tilde{T}_2(x, y)$ is a holomorphic function that vanishes at the origin and whose derivative also vanishes at the origin. Hence, by the implicit function theorem, we can also write $y$ as a function of $z$ and $x$, i.e.

$$y = z + x + T_2(x, z)$$

where $T_2(x, z)$ is a holomorphic function that vanishes at the origin and whose derivative also vanishes at the origin.

Let us denote the partials with respect to the old coordinates as $\rho_{ij}$ and the partials with respect to the new coordinates as $\rho'_{ij}$. Our function is then given by

$$\rho = A_0(x) + A_1(x)y + A_2(x)y^2 + \ldots$$

in terms of the old and new coordinates, respectively. Note that in the new coordinates,

$$\rho'_{00}, \rho'_{10}, \rho'_{01} = 0, \quad \rho'_{02} \neq 0.$$  

Now, it is easy to see that in both the cases we can make a change of coordinates $y = y_1 + B(x)$ and $z = z_1 + \tilde{B}(x)$ so that the function is given by

$$\rho = \hat{A}_0(x) + \hat{A}_2(x)y_1^2 + \ldots$$

where $T_2(x, z)$ is a holomorphic function that vanishes at the origin and whose derivative also vanishes at the origin.

We will now give the proof of Corollary 1.4.7 and 1.4.12. In particular, this will show that the sections we have obtained are well defined on the quotient space (in addition to being well defined under a change of coordinates/curves and change of trivializations).

In all these discussions $T_1$, $T_2$, $\tilde{T}_1$ and $\tilde{T}_2$, will denote holomorphic functions of two variables, that vanish at the origin and whose derivatives also vanish at the origin.

Detailed proof of Corollary 1.4.7

Recall that Corollary 1.4.7 claims that the induced sections $\psi_{\mathcal{P}A_k}$ are well defined. Let $\rho : \mathbb{C}^2 \rightarrow \mathbb{C}$ be a holomorphic function with the following properties:

$$\rho_{00}, \rho_{10}, \rho_{01} = 0, \quad \rho_{02} \neq 0.$$  

Initially our coordinates are $x$ and $y$. The new coordinates are $x$ and $z = y - x - \tilde{T}_2(x, y)$, where $\tilde{T}_2(x, y)$ is a holomorphic function that vanishes at the origin and whose derivative also vanishes at the origin. Hence, by the implicit function theorem, we can also write $y$ as a function of $z$ and $x$, i.e.

$$y = z + x + T_2(x, z)$$

where $T_2(x, z)$ is a holomorphic function that vanishes at the origin and whose derivative also vanishes at the origin.

Let us denote the partials with respect to the old coordinates as $\rho_{ij}$ and the partials with respect to the new coordinates as $\rho'_{ij}$. Our function is then given by

$$\rho = A_0(x) + A_1(x)y + A_2(x)y^2 + \ldots$$

in terms of the old and new coordinates, respectively. Note that in the new coordinates,

$$\rho'_{00}, \rho'_{10}, \rho'_{01} = 0, \quad \rho'_{02} \neq 0.$$  

Now, it is easy to see that in both the cases we can make a change of coordinates $y = y_1 + B(x)$ and $z = z_1 + \tilde{B}(x)$ so that the function is given by

$$\rho = \hat{A}_0(x) + \hat{A}_2(x)y_1^2 + \ldots$$

This just follows from implicit function theorem, since $\rho_{02} \neq 0$ and $\rho'_{02} \neq 0$. (See the proof of Lemma 1.4.5). The fact that $\rho_{11}$ and $\rho_{20}$ are equal to 0 is never really used to get the existence of $B(x))$. Now suppose that

$$\hat{A}_0(x) = \alpha_2(\rho_{ij})x^2 + \alpha_3(\rho_{ij})x^3 + \ldots$$

A little bit of thought will reveal that as functionals, $\alpha_2 = \beta_2$, $\alpha_3 = \beta_3$ and so on. This because the procedure to obtain the $\beta_k$ is exactly the same as with $\alpha_k$, except we are just replacing the $\rho_{ij}$ with $\rho'_{ij}$. Of course, as a function of $\rho_{ij}$, the coefficient of $x^k$ in $\hat{A}_0(x)$ is a priori completely different from the coefficient of $x^k$ in $A_0(x)$. Now we make an important claim:
Claim 1.5.1. As functions of $x$ we have the following equality:

$$
\hat{A}_0(x) = \hat{A}_0(x).
$$

Before proving the claim, let us explain the consequence of this claim. Combined with the observation that $\alpha_k = \beta_k$ as functionals, the claim implies that

$$
\alpha_k(\rho_{ij}) = \alpha_k(\rho'_{ij}) \quad \forall k.
$$

In particular, this implies that the quantity $\alpha_k(\rho_{ij})$ is invariant under $y$ going to $y + x + T_2(x, y)$. Furthermore, if we assume in addition that $\rho_{11}, \rho_{20} = 0$ then the quantities $\alpha_k(\rho_{ij})$ are the quantities $A^\rho_k$ obtained in (1.4.3).

Proof of Claim 1.5.1: First we claim that

$$
B(x) = \tilde{B}(x) + x + T_2(x, \tilde{B}(x)).
$$

In order to prove that, let

$$
\Phi(x, y) := \frac{\partial \rho(x, y)}{\partial y} \quad \text{and} \quad \tilde{\Phi}(x, z) := \frac{\partial \rho(x, z + x + T_2(x, z))}{\partial z}.
$$

By the chain rule

$$
\tilde{\Phi}(x, z) = \Phi(x, z + x + T_2(x, z)).
$$

Hence,

$$
0 = \tilde{\Phi}(x, \tilde{B}(x)) \quad \text{by definition of } \tilde{B}(x).
$$

$$
= \Phi(x, \tilde{B}(x) + x + T_2(x, \tilde{B}(x))) \quad \text{using equation (1.5.15)}
$$

But we also have that there exist a unique solution $B(x)$ with $B(0) = 0$ such that

$$
\Phi(x, B(x)) = 0
$$

(by the implicit function theorem). Hence

$$
B(x) = \tilde{B}(x) + x + T_2(x, \tilde{B}(x)).
$$

Next we recall that our function originally is given by $\rho(x, y)$ (as a function of $x$ and $y$) and then it is written as $\rho(x, z + x + T_2(x, z))$ (as a function of $z$ and $x$). Hence, observe that:

$$
\hat{A}_0(x) = \rho(x, B(x)) \quad (1.5.15)
$$

$$
\hat{A}_0(x) = \rho(x, \tilde{B}(x) + x + T_2(x, \tilde{B}(x))) \quad (1.5.16)
$$

But we just showed that $\tilde{B}(x) + x + T_2(x, \tilde{B}(x)) = B(x)$. Hence:

$$
\hat{A}_0(x) = \hat{A}_0(x). \quad (1.5.17)
$$

This proves the claim. Hence, in particular the quantities $A^\rho_k$ are all invariant under $y$ going to $y + x + T_2(x, y)$. $\square$
Next we will keep $y$ same and show that if $A^\rho_i = 0$ for all $i \leq k$, then $A^\rho_{k+1}$ is invariant under $x \rightarrow x + T_1(x,y)$. Again, we assume that $\rho : \mathbb{C}^2 \rightarrow \mathbb{C}$ be a holomorphic function with the following properties:

$$\rho_{00}, \rho_{10}, \rho_{01} = 0, \quad \rho_{02} \neq 0.$$ 

The old coordinates are $x$ and $y$; the new coordinates are $w := x - \tilde{T}_1(x,y)$ and $y$. By the implicit function theorem, we get that

$$x = w + T_1(w,y).$$

As before, we denote the partials with respect to the old coordinates as $\rho_{ij}$ and the partials with respect to the new coordinates as $\rho'_{ij}$. Note that

$$\rho'_{00}, \rho'_{10}, \rho'_{01} = 0, \quad \rho'_{02} \neq 0.$$ 

Now, it is easy to see that in both the cases we can make a change of coordinates $y = y_1 + B(x)$ and $y = \tilde{y}_1 + \tilde{B}(w)$ so that the function is given by

$$\rho = \hat{A}_0(x) + \hat{A}_2(x)y_1^2 + \ldots$$

$$= \hat{A}_0(w) + \hat{A}_2(w)\tilde{y}_1^2 + \ldots$$

This just follows from implicit function theorem (use $\rho_{02} \neq 0$ and $\rho'_{02} \neq 0$). It is easy to see that we obtain the second presentation of the function (in terms of $w$ and $\tilde{y}_1$) from the first presentation (in terms of $x$ and $y_1$) by making the following replacement

$$x \rightarrow w + T_1(w, \tilde{y}_1 + \hat{A}(w))$$

$$y_1 \rightarrow \tilde{y}_1 + \hat{A}(w) - A(w + T_1(w, \tilde{y}_1 + \hat{A}(w)))$$

Now we claim that

$$\tilde{A}(w) - A(w + T_1(w, \hat{A}(w))) = 0$$

Before proving (1.5.20), let us explain the consequence of that equation. When we make the replacement of $y_1$ as given in equation (1.5.19), there is no pure $w$ term. Hence, the coefficients of $w^n$ in $\hat{A}_0(w)$ are not affected by this replacement; it is only affected by the replacement of $x$. Finally, a little bit of thought now reveals that the germ of $\hat{A}_0(w)$ is the same as the germ of $\hat{A}_0(x)$. (The smallest power of $x$ that has non zero coefficient is the same as the smallest power of $w$ that has non zero coefficient. Moreover, that coefficient is also the same.) In particular, if the $A^\rho_i$ are zero for all $i \leq k$, then $A^\rho_{k+1}$ is invariant under $x$ going to $x + T_1(x,y)$.

In order to prove equation (1.5.20), let

$$\Phi(x,y) := \frac{\partial \rho(x,y)}{\partial y} \quad \text{and} \quad \tilde{\Phi}(w,y) := \frac{\partial \rho(w + T_1(w,y),y)}{\partial y}.$$

By the chain rule

$$\tilde{\Phi}(w,y) = \Phi(w + T_1(w,y),y).$$
Hence,

\[ 0 = \tilde{\Phi}(w, \tilde{B}(w)) \]
\[ = \Phi(w + T_1(w, \tilde{B}(w)), \tilde{B}(w)) \]

Next we note that

\[ \Phi(x, B(x)) = 0 \quad \forall x \]
\[ \implies \Phi(w + T_1(w, \tilde{B}(w)), B(w + T_1(w, \tilde{B}(w)))) = 0 \]
\[ \implies B(w + T_1(w, \tilde{B}(w))) = \tilde{B}(w) \]

which proves (1.5.20).

Finally, it is also very easy to see that if \( A_i^0 = 0 \) for all \( i \leq k \), and we multiply \( \rho(x, y) \) by a non zero function \( \tau(x, y) \), then \( A_{k+1}^0 \) gets multiplied by \( \tau(0, 0) \). This has already been shown in the short proof of Corollary 1.4.7. This concludes our proof of Corollary 1.4.7. \( \square \)

**Detailed proof of Corollary 1.4.12**

Recall that we are trying to show that the induced sections \( \psi_{\mathcal{P}D_1} \) are well defined. Let \( \rho : \mathbb{C}^2 \rightarrow \mathbb{C} \) be a holomorphic function with the following properties:

\( \rho_{00}, \rho_{10}, \rho_{01}, \rho_{20}, \rho_{11}, \rho_{02} = 0, \quad \rho_{12} \neq 0. \)

Initially our coordinates are \( x \) and \( y \). The new coordinates are \( x \) and \( z = y - x - \tilde{T}_2(x, y) \), where \( \tilde{T}_2(x, y) \) is a holomorphic function that vanishes at the origin and whose derivative also vanishes at the origin. Hence, by the implicit function theorem, we can also write \( y \) as a function of \( z \) and \( x \), i.e.

\[ y = z + x + T_2(x, z) \]

where \( T_2(x, z) \) is a holomorphic function that vanishes at the origin and whose derivative also vanishes at the origin.

Let us denote the partials with respect to the old coordinates as \( \rho_{ij} \) and the partials with respect to the new coordinates as \( \rho'_{ij} \). Note that

\[ \rho'_{00}, \rho'_{10}, \rho'_{01}, \rho'_{20}, \rho'_{11}, \rho'_{02} = 0, \quad \rho'_{12} \neq 0. \]

Now, it is easy to see that in both the cases we can make a change of coordinates \( x = x_1 + G(y) \) and \( x = x_1 + \bar{G}(z) \) so that the function is given by

\[ \rho = x_1 g(x_1, y) \]
\[ = x_1 \hat{g}(x_1, z) \]

To see why this is so, look at the end of the proof of Lemma 1.4.11, where we show the existence of \( G(y) \). It is easy to see that the conditions \( \rho_{30} = 0 \) and \( \rho_{21} = 0 \) were not really needed to get the existence of \( G \).

Next, it is easy to see that in both cases, we can make a change of coordinate \( y = y_1 + B(x_1) \) and \( z = z_1 + \bar{B}(x_1) \) so that the function is given by

\[ \rho = x_1 (A_0(x_1) + \hat{A}_2(x_1)y_1^2 + \ldots) \]
\[ = x_1 (\hat{A}_0(x_1) + \hat{A}_2(x_1)z_1^2 + \ldots) \]
This just follows from implicit function theorem (use \( \rho_{12} \neq 0 \) and \( \rho'_{12} \neq 0 \)). Now suppose that
\[
\hat{A}_0(x_1) = \alpha_2(\rho_{ij})x_1^2 + \alpha_3(\rho_{ij})x_1^3 + \ldots \\
\hat{\hat{A}}_0(x_1) = \beta_2(\rho'_{ij})x_1^2 + \beta_3(\rho'_{ij})x_1^3 + \ldots
\]
\begin{equation}
(1.5.22)
\end{equation}

A little bit of thought will reveal that as functionals, \( \alpha_2 = \beta_2, \alpha_3 = \beta_3 \) and so on. This is because the procedure to obtain the \( \beta_k \) is exactly the same as with \( \alpha_k \), except we are just replacing the \( \rho_{ij} \) with \( \rho'_{ij} \). Now we make an important claim:

**Claim 1.5.2.** As functions of \( x_1 \) we have the following equality:
\[
\hat{A}_0(x_1) = \hat{\hat{A}}_0(x_1).
\]

Before proving the claim, let us explain the consequence of this claim. Combined with the observation that \( \alpha_k = \beta_k \) as functionals, the claim implies that
\[
\alpha_k(\rho_{ij}) = \alpha_k(\rho'_{ij}) \quad \forall k.
\]

In particular, this implies that the quantity \( \alpha_k(\rho_{ij}) \) is invariant under \( y \) going to \( y + x + T_2(x,y) \).

Furthermore, if we assume in addition that \( \rho_{31}, \rho_{21} = 0 \) then the quantities \( \alpha_k(\rho_{ij}) \) are the sections \( D_k^0 \) obtained in \( (1.4.10) \). Hence, \( D_k^0 \) are invariant under \( y \) going to \( y + x + T_2(x,y) \).

The proof of the claim 1.5.2 is identical to the proof of claim 1.5.1. \( \square \)

Next, as before it is easy to see that if \( D_k^0 = 0 \) for all \( i \leq k \), then \( D_{k+1}^0 \) is invariant under \( x \mapsto x + T_1(x,y) \), using a similar argument as before.

Finally, it is also very easy to see that if \( D_k^0 = 0 \) for all \( i \leq k \), and we multiply \( \rho(x,y) \) by a non-zero function \( \tau(x,y) \), then \( D_{k+1}^0 \) gets multiplied by \( \tau(0,0) \). This has already been shown in the short proof of Corollary 1.4.12. Combined with all this, we obtain a detailed proof of Corollary 1.4.12. \( \square \)

### 1.6 Transversality

In this section we give an explicit description of the spaces \( A_0, A_1 \) and \( P\mathcal{X}_k \) in terms of vanishing and non-vanishing of bundle sections. Before proceeding, let us state three important Lemmas. We will then show that these spaces satisfy the hypothesis of one of these three Lemmas.

**Lemma 1.6.1.** Let \( M \) be a smooth manifold and \( \{S_i\}_{i=0}^k \) be a family of subspaces defined as:
\[
S_i := \{ p \in M : \zeta_0(p) = 0, \ldots, \zeta_i(p) = 0, \zeta_{i+1}(p) \neq 0 \} \quad \forall \ 0 \leq i \leq k
\]
\begin{equation}
(1.6.1)
\end{equation}

where \( \zeta_i : M \rightarrow V_i \) are sections of vector bundles only defined on the subspace
\[
\{ p \in M : \zeta_0(p), \ldots, \zeta_{i-1}(p) = 0 \}.
\]

If the section
\[
\zeta_i : \{ p \in M : \zeta_0(p), \ldots, \zeta_{i-1}(p) = 0 \} \rightarrow V_i
\]
is transverse to the zero set for \( 0 \leq i \leq k + 1 \) then
\[
\overline{S}_{k-1} = \{ p \in M : \zeta_0(p) = 0, \ldots, \zeta_{k-1}(p) = 0 \} = S_{k-1} \cup \overline{S}_k
\]
\begin{equation}
(1.6.2)
\end{equation}

In particular, \( \overline{S}_{k-1} \) is a smooth manifold.
Lemma 1.6.2. Let $M$ be a smooth manifold and \( \{ \mathcal{S}_i \}_{i=1}^{k} \) be a family of subspaces defined as follows:
\[
\mathcal{S}_{i-1} = \{ p \in M : \zeta_0(p) \neq 0, \varphi(p) \neq 0 \}, \\
\mathcal{S}_i = \{ p \in M : \zeta_0(p) = 0, \ldots, \zeta_i(p) = 0, \zeta_{i+1}(p) \neq 0, \varphi(p) \neq 0 \} \quad \forall \ 0 \leq i \leq k
\]
where $\zeta_i : \rightarrow V_i$ are sections of vector bundles only defined on the subspace
\[
\{ p \in M : \zeta_0(p), \ldots, \zeta_{i-1}(p) = 0 \}
\]
and $\varphi : M \rightarrow W$ is defined. Suppose that
\[
\zeta_i : \{ p \in M : \zeta_0(p) = 0, \ldots, \zeta_{i-1}(p) = 0, \varphi(p) \neq 0 \} \rightarrow V_i
\]
is transverse to the zero set for $0 \leq i \leq k+1$. Then
\[
\overline{\mathcal{S}}_{k-1} = \mathcal{S}_{k-1} \cup \overline{\mathcal{S}}_k \cup \mathcal{B} \quad \text{where} \quad \mathcal{B} := \{ p \in \overline{\mathcal{S}}_{k-1} : \varphi(p) = 0 \}. \quad (1.6.3)
\]
Furthermore, if $\varphi : M \rightarrow W$ is transverse to the zero set, then
\[
M = \mathcal{S}_{-1} \cup \overline{\mathcal{S}}_0 \cup \mathcal{B}' \quad \text{where} \quad \mathcal{B}' := \{ p \in M : \varphi(p) = 0 \}. \quad (1.6.4)
\]

Lemma 1.6.3. Let $M$ be a smooth manifold and let $\mathcal{S}_0 \subset M$ be a subspace defined by
\[
\mathcal{S}_0 = \{ p \in M : \zeta_0(p) = 0, \zeta_1(p) \neq 0, \varphi(p) \neq 0 \}.
\]
Here $\zeta_i : M \rightarrow V_i$ are sections of vector bundles only defined on the subspace
\[
\{ p \in M : \zeta_0(p), \ldots, \zeta_{i-1}(p) = 0 \}
\]
and $\varphi : M \rightarrow W$ is defined. If the sections
\[
\zeta_0 : M \rightarrow V_0, \quad \varphi : \zeta_0^{-1}(0) \rightarrow W, \quad \zeta_1 : \zeta_0^{-1}(0) - \varphi^{-1}(0) \rightarrow V_1
\]
are transverse to the zero set, then
\[
\overline{\mathcal{S}}_0 = \{ p \in M : \zeta_0(p) = 0 \}. \quad (1.6.5)
\]
In particular, $\overline{\mathcal{S}}_0$ is a smooth manifold.

Proof of Lemma 1.6.1: It is evident that the left hand side of (1.6.1) is a subset of its right hand side, since all the sections are continuous. For the converse, we will show that if $p$ belongs to the right hand side, then there exists a sequence $p_n$ in $\mathcal{S}_{k-1}$ that converges to $p$. Since the section
\[
\zeta_k : \{ p \in M : \zeta_0(p) = 0, \ldots, \zeta_{k-1}(p) = 0 \} \rightarrow V_k
\]
is transverse to the zero set, there exists a solution $p_t$ near $p$ to the set of equations
\[
\zeta_0(p_t) = 0, \ldots, \zeta_{k-1}(p_t) = 0, \zeta_k(p_t) = t
\]
if $t$ is sufficiently small. By definition, $p_t$ belongs to $\mathcal{S}_{k-1}$ if $t \neq 0$. This gives us a sequence that lies in $\mathcal{S}_{k-1}$ and converges to $p$. To finish the proof, it suffices to prove (1.6.2) by showing that
\[
\overline{\mathcal{S}}_k = \{ p \in M : \zeta_0(p) = 0, \ldots, \zeta_{k-1}(p) = 0, \zeta_k(p) = 0 \}
\]
This follows from an identical argument as before, using transversality of the section
\[ \zeta_{k+1} : \{ p \in M : \zeta_0(p) = 0, \ldots, \zeta_k(p) = 0 \} \longrightarrow V_{k+1}. \]

**Proof of Lemma 1.6.2:** We will show that the left hand side of (1.6.3) is a subset of its right hand side. We may assume that \( p \in \bar{S}_{k-1} - S_{k-1} \) and \( \varphi(p) \neq 0 \). We claim that \( p \in \bar{S}_k \). In other words, we need to show that there exists a sequence in \( S_k \) that converges to \( p \). Since the section
\[ \zeta_{k+1} : \{ p \in M : \zeta_0(p) = 0, \ldots, \zeta_k(p) = 0, \varphi(p) \neq 0 \} \longrightarrow V_{k+1} \]
is transverse to the zero set, there exists a solution \( p_t \) near \( p \) to the set of equations
\[ \zeta_0(p_t) = 0, \ldots, \zeta_k(p_t) = 0, \zeta_{k+1}(p_t) = t \]
if \( t \) is sufficiently small. By definition, \( p_t \) belongs to \( S_k \) if \( t \neq 0 \). This gives us a sequence that lies in \( S_k \) and converges to \( p \). This proves that the left hand side of (1.6.3) is a subset of its right hand side; if \( p \in \bar{S}_{k-1} - S_{k-1} \) and \( \varphi(p) = 0 \), then \( p \in \mathcal{B} \). Next, let us show that the right hand side of (1.6.3) is a subset of its left hand side. Since \( \mathcal{B} \subset \bar{S}_{k-1} \), it suffices to show that \( \bar{S}_k \subset \bar{S}_{k-1} \). We need to show that if \( p \in \bar{S}_k \), then there exists a sequence in \( S_{k-1} \) that converges to \( p \). Since by hypothesis \( p \in \bar{S}_k \), there exists a sequence \( p_n \in S_k \) that converges to \( p \), i.e., \( p_n \) satisfies the equations:
\[ \zeta_0(p_n) = 0, \ldots, \zeta_{k-1}(p_n) = 0, \zeta_k(p_n) = 0, \zeta_{k+1}(p_n) \neq 0, \varphi(p_n) \neq 0. \]
However, since the section
\[ \zeta_k : \{ p \in M : \zeta_0(p) = 0, \ldots, \zeta_{k-1}(p) = 0, \varphi(p) \neq 0 \} \longrightarrow V_k \]
is transverse to the zero set, we can conclude that there exists a solution \( p_{n,t} \) near \( p_n \) for the set of equations:
\[ \zeta_0(p_{n,t}) = 0, \ldots, \zeta_{k-1}(p_{n,t}) = 0, \zeta_k(p_{n,t}) = t. \]
Moreover, since \( \varphi(p_n) \neq 0 \) and \( \varphi \) is continuous, we get that \( \varphi(p_{n,t}) \neq 0 \) if \( t \) is sufficiently small. Hence, \( p_{n,t} \) lies in \( S_{k-1} \). This gives us a sequence in \( S_{k-1} \) that converges to \( p \), which proves (1.6.3). Finally, we will prove (1.6.4). Let us prove that
\[ \bar{S}_0 \supset \{ p \in M : \zeta_0(p) = 0, \varphi(p) \neq 0 \}. \] (1.6.6)
In other words, we need to show that if \( \zeta_0(p) = 0 \) and \( \varphi(p) \neq 0 \), then there exists a sequence in \( S \) that converges to \( p \). Since the section
\[ \zeta_1 : \{ p \in M : \zeta_0(p) = 0, \varphi(p) \neq 0 \} \longrightarrow V_1 \]
is transverse to the zero set, there exists a solution \( p_t \) near \( p \) to the set of equations
\[ \zeta_0(p_t) = 0, \zeta_1(p_t) = t. \]
Since \( \varphi \) is continuous and \( \varphi(p) \neq 0 \), we get that \( \varphi(p_t) \neq 0 \) if \( t \) is sufficiently small. This gives us the desired sequence in \( S_0 \). Now using (1.6.6) and the definition of \( S_{-1} \) and \( \mathcal{B} \), we get that
\[ M = \{ p \in M : \zeta_0(p) \neq 0, \varphi(p) \neq 0 \} \cup \{ p \in M : \zeta_0(p) = 0, \varphi(p) \neq 0 \} \cup \{ p \in M : \varphi(p) = 0 \} \]
is contained in $\mathcal{S}_1 \cup \mathcal{S}_0 \cup B'$. The reverse inclusion is vacuous. □

**Proof of Lemma 1.6.3:** Observe that the left hand side of (1.6.5) is a subset of its right hand side. For the converse, assume that $p$ belongs to the right hand side. We need to show that there exists a sequence in $\mathcal{S}_0$ that converges to $p$. Let us assume that $\varphi(p) \neq 0$. Since the section

$$\zeta_1 : \{ p \in M : \zeta_0(p) = 0, \varphi(p) \neq 0 \} \rightarrow V_1$$

is transverse to the zero set, there exists a solution $p_t$ near $p$ to the set of equations $\zeta_0(p_t) = 0$ and $\zeta_1(p_t) = t$. Moreover, since $\varphi(p) \neq 0$, we conclude that $\varphi(p_t) \neq 0$ if $t$ is small. Hence, $p_t \in \mathcal{S}_0$ for all small $t$ which gives us the desired sequence. Next let us assume that $\varphi(p) = 0$. Since the section

$$\varphi : \{ p \in M : \zeta_0(p) = 0 \} \rightarrow W$$

is transverse to the zero set, there exists a solution $p_{t_1}$ near $p$ to the set of equations

$$\zeta_0(p_{t_1}) = 0, \varphi(p_{t_1}) = t_1.$$ 

Since $\varphi(p_{t_1}) \neq 0$, there exists a solution $p_{t_1,t_2}$ near $p_{t_1}$ to the set of equations $\zeta_0(p_{t_1,t_2}) = 0$ and $\zeta_1(p_{t_1,t_2}) = t_2$ because the section

$$\zeta_1 : \{ p \in M : \zeta_0(p) = 0, \varphi(p) \neq 0 \} \rightarrow V_1$$

is transverse to the zero set. Moreover, since $\varphi(p_{t_1}) \neq 0$, we infer that $\varphi(p_{t_1,t_2}) \neq 0$ if $t_2$ is sufficiently small. Hence, $p_{t_1,t_2}$ lies in $\mathcal{S}_0$. This gives us the desired sequence which proves Lemma 1.6.3. □

### 1.6.1 Description of spaces as algebraic varieties

We will now give an explicit description of our spaces as algebraic varieties. Although we are going to state a large number of propositions, the statement of these propositions and their proofs are very similar in nature. Moreover it makes it much easier to read the remainder of the document (section 1.7 onwards). Hence we have decided to organized this section in this particular way. The reader can refer to section 1.3 to recapitulate all the definitions and notation.

**Proposition 1.6.4.** The space $\mathcal{A}_0$ can be described as

$$\mathcal{A}_0 = \{(\bar{f}, \bar{p}) \in \mathcal{D} \times \mathbb{P}^2 : \psi_{\mathcal{A}_0}(\bar{f}, \bar{p}) = 0, \psi_{\mathcal{A}_1}(\bar{f}, \bar{p}) \neq 0\}.$$  

(1.6.7)

Furthermore, the sections of the vector bundles

$$\psi_{\mathcal{A}_0} : \mathcal{D} \times \mathbb{P}^2 \rightarrow \mathcal{L}_{\mathcal{A}_0}, \psi_{\mathcal{A}_1} : \psi_{\mathcal{A}_0}^{-1}(0) \rightarrow \mathcal{V}_{\mathcal{A}_1}$$

are transverse to the zero set. In particular, $\mathcal{A}_0$ is a smooth manifold of dimension $\delta_d + 1$.

**Corollary 1.6.5.** The space $\overline{\mathcal{A}_0}$ is a smooth manifold of dimension $\delta_d + 1$ that can be described as

$$\overline{\mathcal{A}_0} = \{(\bar{f}, \bar{p}) \in \mathcal{D} \times \mathbb{P}^2 : \psi_{\mathcal{A}_0}(\bar{f}, \bar{p}) = 0\}.$$ 

**Proposition 1.6.6.** The space $\mathcal{A}_1$ can be described as

$$\mathcal{A}_1 = \{(\bar{f}, \bar{p}) \in \overline{\mathcal{A}_0} : \psi_{\mathcal{A}_1}(\bar{f}, \bar{p}) = 0, \psi_{\mathcal{A}_2}(\bar{f}, \bar{p}) \neq 0, \psi_{\mathcal{D}_1}(\bar{f}, \bar{p}) \neq 0\}.$$  

(1.6.8)
Furthermore, the sections of the vector bundles,

\[ \psi_{A_1} : A_0 \rightarrow V_{A_1}, \psi_{D_4} : \psi_{A_1}^{-1}(0) \rightarrow V_{D_4}, \psi_{A_2} : \psi_{A_1}^{-1}(0) - \psi_{D_4}^{-1}(0) \rightarrow L_{A_2} \]

are transverse to the zero set if \( d \geq 2^7 \). In particular, \( A_1 \) is a smooth manifold of dimension \( \delta_d - 1 \) if \( d \geq 1 \).

**Corollary 1.6.7.** The space \( \overline{A}_1 \) is a smooth manifold of dimension \( \delta_d - 1 \) that can be described as

\[ \overline{A}_1 = \{ (f, \tilde{p}) \in A_0 : \psi_{A_1}(f, \tilde{p}) = 0 \}, \quad \text{provided } d \geq 2. \]

**Corollary 1.6.8.** The spaces \( A_2 \) and \( D_4 \) are smooth manifolds, of dimension \( \delta_d - 2 \) and \( \delta_d - 4 \) respectively, if \( d \geq 2 \).

**Proposition 1.6.9.** The space \( \hat{A}_0 \) can be described as

\[ \hat{A}_0 = \{ (f, l_{\tilde{p}}) \in D \times \mathbb{P}^2 : \Psi_{\hat{A}_0}(f, l_{\tilde{p}}) = 0, \Psi_{\hat{A}_1}(f, l_{\tilde{p}}) \neq 0 \}. \]

Furthermore, the sections of the vector bundles

\[ \Psi_{\hat{A}_0} : D \times \mathbb{P}^2 \rightarrow L_{\hat{A}_0}, \Psi_{\hat{A}_1} : \Psi_{\hat{A}_0}^{-1}(0) \rightarrow V_{A_1} \]

are transverse to the zero set.

**Corollary 1.6.10.** The space \( \overline{A}_0 \) is a smooth manifold of dimension \( \delta_d + 2 \) that can be described as

\[ \overline{A}_0 = \{ (f, l_{\tilde{p}}) \in D \times \mathbb{P}^2 : \Psi_{\overline{A}_0}(f, l_{\tilde{p}}) = 0 \}. \]

**Proposition 1.6.11.** The space \( \hat{A}_1 \) can be described as

\[ \hat{A}_1 = \{ (f, l_{\tilde{p}}) \in \overline{A}_0 : \Psi_{\hat{A}_1}(f, l_{\tilde{p}}) = 0, \Psi_{\hat{A}_2}(f, l_{\tilde{p}}) \neq 0, \Psi_{D_4}(f, l_{\tilde{p}}) \neq 0 \}. \]

Furthermore, the sections of the vector bundles

\[ \Psi_{\hat{A}_1} : \overline{A}_0 \rightarrow V_{A_1}, \Psi_{D_4} : \Psi_{\hat{A}_1}^{-1}(0) \rightarrow V_{D_4}, \Psi_{A_2} : \Psi_{\hat{A}_1}^{-1}(0) - \Psi_{D_4}^{-1}(0) \rightarrow L_{A_2} \]

are transverse to the zero set if \( d \geq 2^8 \).

**Corollary 1.6.12.** The space \( \overline{A}_1 \) is a smooth manifold of dimension \( \delta_d \) that can be described as

\[ \overline{A}_1 = \{ (f, l_{\tilde{p}}) \in \overline{A}_0 : \Psi_{\overline{A}_1}(f, l_{\tilde{p}}) = 0 \}, \text{ provided } d \geq 2. \]

**Proposition 1.6.13.** The space \( \hat{A}_1^\# \) can be described as

\[ \hat{A}_1^\# = \{ (f, l_{\tilde{p}}) \in D \times \mathbb{P}^2 : \Psi_{\hat{A}_0}(f, l_{\tilde{p}}) = 0, \Psi_{\hat{A}_1}(f, l_{\tilde{p}}) = 0, \Psi_{P_{A_2}}(f, l_{\tilde{p}}) \neq 0 \}. \quad (1.6.9) \]

Furthermore, the sections of the vector bundles

\[ \Psi_{\hat{A}_0} : D \times \mathbb{P}^2 \rightarrow L_{\hat{A}_0}, \Psi_{\hat{A}_1} : \Psi_{\hat{A}_0}^{-1}(0) \rightarrow V_{A_1}, \Psi_{P_{A_2}} : \Psi_{\hat{A}_1}^{-1}(0) \rightarrow V_{P_{A_2}} \]

are transverse to the zero set, provided \( d \geq 2 \).

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\(^7\) The section \( \psi_{D_4}(f, \tilde{p}) \) is non-zero is vacuously true if \( \psi_{A_1}(f, \tilde{p}) \neq 0 \). We have stated the proposition in this way so that it is clear that our spaces satisfy the hypothesis of Lemma 1.6.3.

\(^8\) Again, \( \Psi_{D_4}(f, l_{\tilde{p}}) \neq 0 \) is vacuously true if \( \Psi_{A_2}(f, l_{\tilde{p}}) \neq 0 \).
Corollary 1.6.14. The space $\overline{A}_1^f$ is a smooth manifold of dimension $\delta_d$ that can be described as

$$\overline{A}_1^f = \{(\tilde{f}, l_p) \in \overline{A}_0 : \Psi_{\overline{A}_1}(\tilde{f}, l_p) = 0\}, \text{ provided } d \geq 2.$$ 

Proposition 1.6.15. The space $\mathcal{P}A_2$ can be described as

$$\mathcal{P}A_2 = \{ (\tilde{f}, l_p) \in \overline{A}_1^f : \Psi_{\mathcal{P}A_2}(\tilde{f}, l_p) = 0, \Psi_{\mathcal{P}A_3}(\tilde{f}, l_p) \neq 0, \Psi_{\mathcal{P}D_4}(\tilde{f}, l_p) \neq 0\}. \quad (1.6.10)$$

Furthermore, the sections of the vector bundles

$$\Psi_{\mathcal{P}A_2} : \overline{A}_1^f \to \mathcal{V}_{\mathcal{P}A_2}, \quad \Psi_{\mathcal{P}A_3} : \Psi_{\mathcal{P}A_2}^{-1}(0) \to \mathcal{L}_{\mathcal{P}A_3}, \quad \Psi_{\mathcal{P}D_4} : \Psi_{\mathcal{P}A_2}^{-1}(0) \to \mathcal{L}_{\mathcal{P}D_4}$$

are transverse to the zero set, provided $d \geq 3$.

Corollary 1.6.16. The space $\overline{P}A_2$ is a manifold of dimension $\delta_d - 2$ and can be described as

$$\overline{P}A_2 = \{ (\tilde{f}, l_p) \in \overline{A}_1^f : \Psi_{\mathcal{P}A_2}(\tilde{f}, l_p) = 0\}, \text{ provided } d \geq 3.$$ 

Proposition 1.6.17. The space $\overline{D}_4^f$ can be described as

$$\overline{D}_4^f = \{(\tilde{f}, l_p) \in D \times \mathbb{P}^2 : \Psi_{\overline{D}_4}^{-1}(\tilde{f}, l_p) = 0, \Psi_{\overline{D}_4}^{-1}(\tilde{f}, l_p) = 0, \Psi_{\overline{D}_4}(\tilde{f}, l_p) = 0, \Psi_{\mathcal{P}A_3}(\tilde{f}, l_p) \neq 0\}. \quad (1.6.11)$$

Furthermore, the sections of the vector bundles

$$\Psi_{\overline{D}_4} : D \times \mathbb{P}^2 \to \mathcal{L}_{\overline{D}_4}, \quad \Psi_{\overline{D}_4}^{-1} : \Psi_{\overline{D}_4}^{-1}(0) \to \mathcal{V}_{\overline{D}_4}, \quad \Psi_{\mathcal{P}A_3} : \Psi_{\overline{D}_4}^{-1}(0) \to \mathcal{L}_{\mathcal{P}A_3}$$

are transverse to the zero set, provided $d \geq 3$.

Corollary 1.6.18. The space $\overline{D}_4^f$ is a manifold of dimension $\delta_d - 4$ and can be described as

$$\overline{D}_4^f = \{(\tilde{f}, l_p) \in D \times \mathbb{P}^2 : \Psi_{\overline{D}_4}(\tilde{f}, l_p) = 0, \Psi_{\overline{D}_4}(\tilde{f}, l_p) = 0, \Psi_{\overline{D}_4}(\tilde{f}, l_p) = 0, \Psi_{\mathcal{P}D_4}(\tilde{f}, l_p) \neq 0\}, \text{ provided } d \geq 3.$$ 

Corollary 1.6.19. The space $\mathcal{P}D_4$ can be described as

$$\mathcal{P}D_4 = \{(\tilde{f}, l_p) \in \overline{P}A_2 : \Psi_{\mathcal{P}A_3}(\tilde{f}, l_p) = 0, \Psi_{\mathcal{P}D_4}(\tilde{f}, l_p) = 0, \Psi_{\mathcal{P}D_4}(\tilde{f}, l_p) \neq 0, \Psi_{\mathcal{P}D_4}(\tilde{f}, l_p) \neq 0\}. \quad (1.6.12)$$

Furthermore, the sections of the vector bundles

$$\Psi_{\mathcal{P}A_3} : \overline{P}A_2 \to \mathcal{L}_{\mathcal{P}A_3}, \quad \Psi_{\mathcal{P}D_4} : \Psi_{\mathcal{P}A_3}^{-1}(0) \to \mathcal{L}_{\mathcal{P}D_4},$$

$$\Psi_{\mathcal{P}D_4} : \Psi_{\mathcal{P}D_4}^{-1}(0) \to \mathcal{L}_{\mathcal{P}D_4}, \quad \Psi_{\mathcal{P}D_4} : \Psi_{\mathcal{P}D_4}^{-1}(0) - \Psi_{\mathcal{P}D_4}^{-1}(0) \to \mathcal{L}_{\mathcal{P}D_4}$$

are transverse to the zero set, provided $d \geq 3$.

Corollary 1.6.20. The space $\overline{PD}_4$ is a manifold of dimension $\delta_d - 3$ and can be described as

$$\overline{PD}_4 = \{(\tilde{f}, l_p) \in \overline{P}A_2 : \Psi_{\mathcal{P}A_3}(\tilde{f}, l_p) = 0, \Psi_{\mathcal{P}D_4}(\tilde{f}, l_p) = 0\}, \text{ provided } d \geq 3.$$
Proposition 1.6.21. The space $\mathcal{P}A_3$ can be described as

$$\mathcal{P}A_3 = \{ (\tilde{f}, l_p) \in \mathcal{P}A_2 : \Psi_{\mathcal{P}A_3}(\tilde{f}, l_p) = 0, \Psi_{\mathcal{P}A_4}(\tilde{f}, l_p) \neq 0, \Psi_{\mathcal{P}D_4}(\tilde{f}, l_p) \neq 0 \}. \quad (1.6.13)$$

Furthermore, the sections of the vector bundles

$$\Psi_{\mathcal{P}A_3} : \mathcal{P}A_2 \rightarrow \mathbb{L}_{\mathcal{P}A_3}, \quad \Psi_{\mathcal{P}D_4} : \Psi_{\mathcal{P}A_3}^{-1}(0) \rightarrow \mathbb{L}_{\mathcal{P}D_4}, \quad \Psi_{\mathcal{P}A_4} : \Psi_{\mathcal{P}A_3}^{-1}(0) - \Psi_{\mathcal{P}D_4}^{-1}(0) \rightarrow \mathbb{L}_{\mathcal{P}A_4}$$

are transverse to the zero set provided $d \geq 4$.

Corollary 1.6.22. The space $\mathcal{P}A_3$ is a manifold of dimension $\delta_d - 3$ and can be described as

$$\mathcal{P}A_3 = \{ (\tilde{f}, l_p) \in \mathcal{P}A_2 : \Psi_{\mathcal{P}A_3}(\tilde{f}, l_p) = 0 \}, \text{ provided } d \geq 4.$$

Corollary 1.6.23. The space $A_3$ is a smooth manifold of dimension $\delta_d - 3$, if $d \geq 3$.

Proposition 1.6.24. If $k > 3$, the space $\mathcal{P}A_k$ can be described as

$$\mathcal{P}A_k = \{ (\tilde{f}, l_p) \in \mathcal{P}A_3 : \Psi_{\mathcal{P}A_k}(\tilde{f}, l_p) = 0 \} \quad (1.6.14)$$

Furthermore, the sections of the vector bundles $\Psi_{\mathcal{P}A_i} : \Psi_{\mathcal{P}A_{i-1}}^{-1}(0) - \Psi_{\mathcal{P}D_4}^{-1}(0) \rightarrow \mathbb{L}_{\mathcal{P}A_i}$ are transverse to the zero set for all $4 \leq i \leq k + 1$, provided $d \geq k + 1$.

Corollary 1.6.25. The space $A_k$ is a smooth manifold of dimension $\delta_d - k$, if $d \geq k$.

Proposition 1.6.26. The space $\mathcal{P}D_5$ can be described as

$$\mathcal{P}D_5 = \{ (\tilde{f}, l_p) \in \mathcal{P}D_4 : \Psi_{\mathcal{P}D_5}(\tilde{f}, l_p) = 0, \Psi_{\mathcal{P}D_6}(\tilde{f}, l_p) \neq 0, \Psi_{\mathcal{P}E_6}(\tilde{f}, l_p) \neq 0 \}. \quad (1.6.15)$$

Furthermore, the sections of the vector bundles

$$\Psi_{\mathcal{P}D_5} : \mathcal{P}D_4 \rightarrow \mathbb{L}_{\mathcal{P}D_5}, \quad \Psi_{\mathcal{P}D_6} : \Psi_{\mathcal{P}D_5}^{-1}(0) \rightarrow \mathbb{L}_{\mathcal{P}D_6}, \quad \Psi_{\mathcal{P}E_6} : \Psi_{\mathcal{P}D_5}^{-1}(0) \rightarrow \mathbb{L}_{\mathcal{P}E_6}$$

are transverse to the zero set, provided $d \geq 4$.

Corollary 1.6.27. The space $\mathcal{P}D_5$ is a manifold of dimension $\delta_d - 5$ and can be described as

$$\mathcal{P}D_5 = \{ (\tilde{f}, l_p) \in \mathcal{P}D_4 : \Psi_{\mathcal{P}D_5}(\tilde{f}, l_p) = 0 \}, \text{ provided } d \geq 4.$$

Proposition 1.6.28. The space $\mathcal{P}D_5'$ can be described as

$$\mathcal{P}D_5' = \{ (\tilde{f}, l_p) \in \mathcal{P}D_4 : \Psi_{\mathcal{P}D_5'}(\tilde{f}, l_p) = 0, \Psi_{\mathcal{P}D_6}(\tilde{f}, l_p) \neq 0, \Psi_{\mathcal{P}D_5}(\tilde{f}, l_p) \neq 0 \}. \quad (1.6.16)$$

Furthermore, the sections of the vector bundles

$$\Psi_{\mathcal{P}D_5'} : \mathcal{P}D_4 \rightarrow \mathbb{L}_{\mathcal{P}D_5'}, \quad \Psi_{\mathcal{P}D_6} : \Psi_{\mathcal{P}D_5'}^{-1}(0) \rightarrow \mathbb{L}_{\mathcal{P}D_6'}, \quad \Psi_{\mathcal{P}D_5} : \Psi_{\mathcal{P}D_5'}^{-1}(0) \rightarrow \mathbb{L}_{\mathcal{P}D_5'}$$

are transverse to the zero set, provided $d \geq 4$.

Corollary 1.6.29. The space $\mathcal{P}D_5'$ is a manifold of dimension $\delta_d - 5$ and can be described as

$$\mathcal{P}D_5' = \{ (\tilde{f}, l_p) \in \mathcal{P}D_4 : \Psi_{\mathcal{P}D_5'}(\tilde{f}, l_p) = 0 \}, \text{ provided } d \geq 4.$$
Corollary 1.6.30. The space $\mathcal{D}_5$ is a smooth manifold of dimension $\delta_d - 5$, if $d \geq 3$.

Proposition 1.6.31. The space $\mathcal{P}\mathcal{E}_6$ can be described as

$$\mathcal{P}\mathcal{E}_6 = \{(\tilde{f}, l_p) \in \mathcal{P}\mathcal{D}_5 : \Psi_{\mathcal{P}\mathcal{E}_6}(\tilde{f}, l_p) = 0, \ \Psi_{\mathcal{P}\mathcal{E}_7}(\tilde{f}, l_p) \neq 0, \ \Psi_{\mathcal{P}\mathcal{X}_6}(\tilde{f}, l_p) \neq 0\} \quad (1.6.17)$$

Furthermore, the sections of the vector bundles

$$\Psi_{\mathcal{P}\mathcal{E}_6} : \mathcal{P}\mathcal{D}_5 \rightarrow \mathbb{L}\mathcal{P}\mathcal{E}_6, \quad \Psi_{\mathcal{P}\mathcal{E}_7} : \mathcal{P}\mathcal{E}_7 \rightarrow \mathbb{L}\mathcal{P}\mathcal{E}_7, \quad \Psi_{\mathcal{P}\mathcal{X}_6} : \mathcal{P}\mathcal{X}_6 \rightarrow \mathbb{L}\mathcal{P}\mathcal{X}_6$$

are transverse to the zero set, provided $d \geq 4$.

Corollary 1.6.32. The space $\mathcal{P}\mathcal{E}_6$ is a manifold of dimension $\delta_d - 6$ and can be described as

$$\mathcal{P}\mathcal{E}_6 = \{(\tilde{f}, l_p) \in \mathcal{P}\mathcal{D}_5 : \Psi_{\mathcal{P}\mathcal{E}_6}(\tilde{f}, l_p) = 0\}, \ \text{provided} \ d \geq 4.$$

Corollary 1.6.33. The space $\mathcal{E}_6$ is a manifold of dimension $\delta_d - 6$, if $d \geq 3$.

Proposition 1.6.34. The space $\mathcal{P}\mathcal{D}_6$ can be described as

$$\mathcal{P}\mathcal{D}_6 = \{(\tilde{f}, l_p) \in \mathcal{P}\mathcal{D}_5 : \Psi_{\mathcal{P}\mathcal{D}_6}(\tilde{f}, l_p) = 0, \ \Psi_{\mathcal{P}\mathcal{D}_7}(\tilde{f}, l) \neq 0, \ \Psi_{\mathcal{P}\mathcal{E}_6}(\tilde{f}, l_p) \neq 0\} \quad (1.6.18)$$

Furthermore, the sections of the vector bundles

$$\Psi_{\mathcal{P}\mathcal{D}_6} : \mathcal{P}\mathcal{D}_5 \rightarrow \mathbb{L}\mathcal{P}\mathcal{D}_6, \quad \Psi_{\mathcal{P}\mathcal{E}_6} : \mathcal{P}\mathcal{E}_6 \rightarrow \mathbb{L}\mathcal{P}\mathcal{E}_6, \quad \Psi_{\mathcal{P}\mathcal{D}_7} : \mathcal{P}\mathcal{D}_7 \rightarrow \mathbb{L}\mathcal{P}\mathcal{D}_7$$

are transverse to the zero set, provided $d \geq 4$.

Corollary 1.6.35. The space $\mathcal{P}\mathcal{D}_6$ is a manifold of dimension $\delta_d - 6$ and can be described as

$$\mathcal{P}\mathcal{D}_6 = \{(\tilde{f}, l_p) \in \mathcal{P}\mathcal{D}_5 : \Psi_{\mathcal{P}\mathcal{D}_6}(\tilde{f}, l_p) = 0\}, \ \text{provided} \ d \geq 4.$$

Corollary 1.6.36. The space $\mathcal{D}_6$ is a manifold of dimension $\delta_d - 6$, if $d \geq 4$.

Proposition 1.6.37. If $k > 6$, then the space $\mathcal{P}\mathcal{D}_k$ can be described as

$$\mathcal{P}\mathcal{D}_k = \{(\tilde{f}, l_p) \in \mathcal{P}\mathcal{D}_5 : \Psi_{\mathcal{P}\mathcal{D}_j}(\tilde{f}, l_p) = 0 \ \text{if} \ 7 \leq j \leq k, \ \Psi_{\mathcal{P}\mathcal{D}_{k+1}}(\tilde{f}, l_p) \neq 0, \ \Psi_{\mathcal{P}\mathcal{E}_6}(\tilde{f}, l_p) \neq 0\} \quad (1.6.20)$$

Furthermore, the section of the vector bundle $\Psi_{\mathcal{P}\mathcal{D}_i} : \mathcal{P}\mathcal{D}_{i-1} \rightarrow \mathbb{L}\mathcal{P}\mathcal{D}_i$ is transverse to the zero set for all $7 \leq i \leq k + 1$ provided $d > k - 2$.

Corollary 1.6.38. The space $\mathcal{D}_k$ is a manifold of dimension $\delta_d - k$, if $d \geq k - 2$.

Proposition 1.6.39. The space $\mathcal{P}\mathcal{E}_7$ can be described as

$$\mathcal{P}\mathcal{E}_7 = \{(\tilde{f}, l_p) \in \mathcal{P}\mathcal{E}_6 : \Psi_{\mathcal{P}\mathcal{E}_7}(\tilde{f}, l_p) = 0, \ \Psi_{\mathcal{P}\mathcal{X}_6}(\tilde{f}, l_p) \neq 0, \ \Psi_{\mathcal{P}\mathcal{X}_8}(\tilde{f}, l_p) \neq 0\} \quad (1.6.21)$$

Furthermore, the sections of the vector bundles

$$\Psi_{\mathcal{P}\mathcal{E}_7} : \mathcal{P}\mathcal{E}_6 \rightarrow \mathbb{L}\mathcal{P}\mathcal{E}_7, \quad \Psi_{\mathcal{P}\mathcal{X}_6} : \mathcal{P}\mathcal{X}_6 \rightarrow \mathbb{L}\mathcal{P}\mathcal{X}_6, \quad \Psi_{\mathcal{P}\mathcal{X}_8} : \mathcal{P}\mathcal{X}_8 \rightarrow \mathbb{L}\mathcal{P}\mathcal{X}_8$$

are transverse to the zero set, provided $d \geq 4$. 

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Corollary 1.6.40. The space $\overline{PE}_7$ is a manifold of dimension $\delta_d - 6$ and can be described as
$$\overline{PE}_7 = \{ (\tilde{f}, l_p) \in \overline{PE}_6 : \Psi_{\overline{PE}_7}(\tilde{f}, l_p) = 0 \}, \text{ provided } d \geq 4.$$ 

Corollary 1.6.41. The space $E_7$ is a manifold of dimension $\delta_d - 7$, if $d \geq 4$.

Proposition 1.6.42. The space $PE_7$ can also be described as
$$PE_7 = \{ (\tilde{f}, l_p) \in \overline{PD}_6 : \Psi_{PE_6}(\tilde{f}, l_p) = 0, \Psi_{PE_6}(\tilde{f}, l_p) \neq 0, \Psi_{P\chi_h}(\tilde{f}, l_p) \neq 0 \}. \quad (1.6.22)$$
Furthermore, the sections of the vector bundles
$$\Psi_{PE_6} : \overline{PD}_6 \to \mathbb{L}_{PE_6}, \quad \Psi_{PE_6} : \Psi_{PE_7}^{-1}(0) \to \mathbb{L}_{PE_6}, \quad \Psi_{P\chi_h} : \Psi_{P\chi_h}^{-1}(0) \to \mathbb{L}_{P\chi_h}$$
are transverse to the zero set, provided $d \geq 4$.

Corollary 1.6.43. The space $\overline{PE}_7$ is a manifold of dimension $\delta_d - 6$ and can also be described as
$$\overline{PE}_7 = \{ (\tilde{f}, l_p) \in \overline{PD}_6 : \Psi_{PE_6}(\tilde{f}, l_p) = 0 \}, \text{ provided } d \geq 4.$$ 

Proposition 1.6.44. The spaces $\hat{X}_8^\#$ and $\hat{X}_8^{\#\#}$ can be described as
$$\hat{X}_8^\# = \{ (\tilde{f}, l_p) \in D \times \mathbb{P}T^2 : \Psi_{\hat{A}_0}(\tilde{f}, l_p) = 0, \Psi_{\hat{A}_1}(\tilde{f}, l_p) = 0, \Psi_{\hat{D}_4}(\tilde{f}, l_p) = 0, \Psi_{\hat{X}_8}(\tilde{f}, l_p) = 0, \Psi_{PE_7}(\tilde{f}, l_p) \neq 0 \}$$
$$\hat{X}_8^{\#\#} = \{ (\tilde{f}, l_p) \in \hat{X}_8^\# : \Psi_{\hat{J}}(\tilde{f}, l_p) \neq 0 \}. \quad (1.6.23)$$
Furthermore, the sections of the vector bundle
$$\Psi_{\hat{A}_0} : D \times \mathbb{P}T^2 \to \mathbb{L}_{\hat{A}_0}, \quad \Psi_{\hat{A}_1} : \Psi_{\hat{A}_0}^{-1}(0) \to \mathbb{V}_{\hat{A}_1}, \quad \Psi_{\hat{D}_4} : \Psi_{\hat{D}_4}^{-1}(0) \to \mathbb{V}_{\hat{D}_4}, \quad \Psi_{\hat{X}_8} : \Psi_{\hat{X}_8}^{-1}(0) \to \mathbb{L}_{\hat{X}_8}, \quad \Psi_{\hat{J}} : \Psi_{\hat{X}_8}^{-1}(0) \to \mathbb{L}_{\hat{J}}$$
are transverse to the zero set, provided $d \geq 4$.

Corollary 1.6.45. The spaces $\overline{X}_8^\#$ and $\overline{X}_8^{\#\#}$ are equal to each other. They are manifolds of dimension $\delta_d - 7$ and can be described as
$$\overline{X}_8^\# = \overline{X}_8^{\#\#} = \{ (f, l_p) \in D \times \mathbb{P}T^2 : \Psi_{\hat{A}_0}(f, l_p) = 0, \Psi_{\hat{A}_1}(f, l_p) = 0, \Psi_{\hat{D}_4}(f, l_p) = 0, \Psi_{\hat{X}_8}(f, l_p) = 0 \}$$
provided $d \geq 4$.

1.6.2 Proofs of Propositions

Let $F \cong \mathbb{C}^{d+1}$ denote the space of homogeneous polynomials of degree $d$. Let $F^*$ be the subspace of non-zero polynomials. This can also be thought of as the space of polynomials in two variables of degree at most $d$. If $V \to M$ is any vector bundle then a section
$$\psi : D \times M \to \pi^*_D \gamma^*_D \otimes \pi^*_MV$$
induces a section
$$\tilde{\psi} : F^* \times M \to \pi^*_MV \quad \text{given by} \quad \tilde{\psi}(f, x) := \{ \psi(\tilde{f}, x) \}(f). \quad (1.6.24)$$
We note that $\psi$ is transverse to zero at $(\tilde{f}, x)$ if and only if $\psi$ is transverse to zero at $(f, x)$.

**Proof of Proposition 1.6.4:** Equation (1.6.7) follows from Corollary 1.4.2. We will now show transversality. We will use the setup of definition 2.3.1, remark 2.3.3 and (2.3.4). Suppose $(\tilde{f}, \tilde{p}) \in \psi_{A_0}^{-1}(0)$. Choose homogeneous coordinates $[X : Y : Z]$ on $\mathbb{P}^2$ so that $\tilde{p} = [0 : 0 : 1]$ and let

$$U := \{ [X : Y : Z] \in \mathbb{P}^2 : Z \neq 0 \}, \quad \varphi_U : U \rightarrow \mathbb{C}^2, \quad \varphi_U([X : Y : Z]) = (X/Z, Y/Z).$$

Let us also denote $x := X/Z$ and $y := Y/Z$. The section

$$\psi_{A_0} : \mathcal{D} \times \mathbb{P}^2 \rightarrow \mathcal{L}_{A_0} := \pi_D^* \gamma_D \otimes \pi_{p_2}^* \gamma_{p_2}^{*}$$

induces a section $\tilde{\psi}_{A_0} : \mathcal{F}^* \times \mathbb{P}^2 \rightarrow \pi_{p_2}^* \gamma_{p_2}^{*}$ as given in (2.3.4). We shall denote the bundle by $\gamma_{p_2}^*$. We now observe that with respect to the standard trivialization of $\gamma_{p_2}^* \rightarrow \mathcal{F}^* \times \mathbb{P}^2$, the induced map $\tilde{\psi}_{A_0} : \mathcal{F}^* \times \mathbb{C}^2 \rightarrow \mathbb{C}$ of the section $\tilde{\psi}_{A_0}$ (cf. (2.3.3), remark 2.3.3) is given by

$$\tilde{\psi}_{A_0}(f; x, y) = f(x, y) := f_{00} + f_{10}x + f_{01}y + \frac{f_{20}}{2}x^2 + f_{11}xy + \frac{f_{02}}{2}y^2 + \cdots.$$

By remark 2.3.3, it suffices to show that this induced map $\tilde{\psi}_{A_0}$ is transverse to zero at $(f, 0, 0)$. Since the Jacobian matrix of this map at $(f, 0, 0)$ is

$$d\tilde{\psi}_{A_0}|_{(f, 0, 0)} = \begin{pmatrix} 1 & 0 & 0 & \cdots \end{pmatrix},$$

where the first column is partial derivative with respect to $f_{00}$, transversality follows. Next we will prove that $\psi_{A_1}$ restricted to $\psi_{A_0}^{-1}(0)$ is transverse to zero. By Lemma 1.4.18, statement 1 we conclude that if $\psi_{A_0}(\tilde{f}, \tilde{p}) = 0$, then $\psi_{A_1}(\tilde{f}, \tilde{p})$ is well defined. Let

$$(\tilde{f}, \tilde{p}) \in \psi_{A_1}^{-1}(0) \subset \psi_{A_0}^{-1}(0).$$

With respect to the standard trivialization of $\pi_D^* \gamma_D \otimes \mathcal{V}_{A_1} \rightarrow \tilde{\psi}_{A_0}^{-1}(0)$, the induced map of the section $\tilde{\psi}_{A_1}$ (cf. definition 2.3.1, (2.3.1)) is given by

$$\tilde{\psi}_{A_1} : (\mathcal{F}^* \times \mathbb{C}^2) \cap \tilde{\psi}_{A_0}^{-1}(0) \rightarrow \mathbb{C}^2, \quad \tilde{\psi}_{A_1}(f; x, y) = (f_x(x, y), f_y(x, y)).$$

Since the function $\tilde{\psi}_{A_0}$ is transverse to the zero set at $(f, 0, 0)$, showing that $\tilde{\psi}_{A_1}$ is transverse to zero at $(f, 0, 0)$ is equivalent to showing that the map

$$\tilde{\psi}_{A_0} \oplus \tilde{\psi}_{A_1} : \mathcal{F}^* \times \mathbb{C}^2 \rightarrow \mathbb{C}^3, \quad \tilde{\psi}_{A_0} \oplus \tilde{\psi}_{A_1}(f; x, y) = (f(x, y), f_x(x, y), f_y(x, y))$$

is transverse to zero. Since $f(x, y) = f_{00} + f_{10}x + f_{01}y + \cdots$, the Jacobian at $(f, 0, 0)$ is

$$d(\tilde{\psi}_{A_0} \oplus \tilde{\psi}_{A_1})|_{(f, 0, 0)} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \end{pmatrix},$$

where the first three columns are partial derivatives with respect to $f_{00}$, $f_{10}$ and $f_{01}$. \qed
Proof of Corollary 1.6.5: This follows immediately from Lemma 1.6.1 and Proposition 1.6.4. □

Proof of Proposition 1.6.6: Equation (1.6.8) follows from Corollary 1.4.4 and Corollary 1.6.5. We have already shown the transversality of \( \bar{\psi}_{A_1} \) in the proof of Proposition 1.6.4. We will now show the transversality of \( \bar{\psi}_{D_4} \) and \( \bar{\psi}_{A_2} \). Let us start with \( \bar{\psi}_{D_4} \). By Lemma 1.4.18, statement 1 we conclude that \( \bar{\psi}_{D_4} \) is well defined restricted to \( \bar{\psi}_{A_1}^{-1}(0) \). As in the proof of Proposition 1.6.4, proving transversality is equivalent to showing that the map

\[
\bar{\psi}_{A_0} \oplus \bar{\psi}_{A_1} \oplus \bar{\psi}_{D_4} : \mathcal{F}^* \times \mathbb{C}^2 \longrightarrow \mathbb{C}^6, \quad (f, x, y) \mapsto (f(x, y), f_x, f_y, f_{xx}, f_{xy}, f_{yy})
\]

is transverse to zero at the point \((f, 0, 0)\). The Jacobian matrix of this map at \((f, 0, 0)\) is a \(6 \times (\delta_d + 3)\) matrix which has full rank if \(d \geq 2\); this follows, for instance, if the first six columns of the matrix are partial derivatives with respect to \(f_0, f_1, f_0, f_{20}, f_{11}\) and \(f_{02}\).

Next let us show transversality of the section \(\bar{\psi}_{A_2}\). The section is well defined on \(\bar{\psi}_{A_1}^{-1}(0)\) by Lemma 1.4.18, statement 1. As before, proving transversality is equivalent to showing that the map

\[
\bar{\psi}_{A_0} \oplus \bar{\psi}_{A_1} \oplus \bar{\psi}_{A_2} : \mathcal{F}^* \times \mathbb{C}^2 \longrightarrow \mathbb{C}^4, \quad (f, x, y) \mapsto (f(x, y), f_x, f_y, f_{xx}f_{yy} - f_{xy}^2)
\]

is transverse to zero at the point \((f, 0, 0)\). The Jacobian matrix of this map at \((f, 0, 0)\) is a \(4 \times (\delta_d + 3)\) matrix which has full rank if \(d \geq 2\) and \(\bar{\psi}_{D_4}(\hat{f}, \tilde{\eta}) \neq 0\) : take the first four columns of the matrix as the partial derivatives with respect to \(f_0, f_1, f_0, f_{20}, f_{02}\) or \(f_{11}\), depending on whether \(f_{02}, f_{20}\) or \(f_{11}\) is non-zero. One of them is guaranteed to be non-zero if \(\bar{\psi}_{D_4}(\hat{f}, \tilde{\eta}) \neq 0\). □

Proof of Corollary 1.6.7: This follows immediately from Lemma 1.6.3 and Proposition 1.6.6. □

Proof of Corollary 1.6.8: This follows from Proposition 1.6.6, Corollary 1.4.7 and Corollary 1.4.10. □

Proof of Proposition 1.6.9: This is identical to the proof of Proposition 1.6.4. □

Proof of Corollary 1.6.10: This follows immediately from Lemma 1.6.1 and Proposition 1.6.9. □

Proof of Proposition 1.6.11: This is identical to the proof of Proposition 1.6.6. □

Proof of Corollary 1.6.12: This follows immediately from Lemma 1.6.3 and Proposition 1.6.11. □

Proof of Proposition 1.6.13: Equation (1.6.9) is the definition of \(\hat{A}_1\), so there is nothing to prove. To show transversality we continue with the setup of the proof of Proposition 1.6.4, but choose coordinate chart so that

\[
\mathcal{U} := \{(a \partial_x, b \partial_y) \in \mathbb{R} \mathbb{P}^2_{\mathcal{U}} : a \neq 0\}, \quad \varphi_{\mathcal{U}} : \mathcal{U} \longrightarrow \mathbb{C}^3, \quad \varphi_{\mathcal{U}}([a \partial_x, b \partial_y]) = (x, y, \eta),
\]

where \(\eta := b/a\). By Lemma 1.4.18, statement 1 we conclude that \(\Psi_{\mathcal{P}A_2}(\hat{f}, \tilde{\eta})\) is well defined restricted to \(\bar{\psi}_{A_1}^{-1}(0)\). With respect to the standard trivialization, the induced map \(\Psi_{\mathcal{P}A_2}\) restricted to \(\bar{\psi}_{A_1}^{-1}(0)\) is given by

\[
\Psi_{\mathcal{P}A_2} : (\mathcal{F}^* \times \mathbb{C}^3) \cap \bar{\psi}_{A_1}^{-1}(0) \longrightarrow \mathbb{C}^2, \quad f, x, y, \eta \mapsto (f_{xx} + \eta f_{xy}, f_{xy} + \eta f_{yy}).
\]
As before, this is equivalent to showing that the map
\[ \Psi_{A_0} \oplus \tilde{\Psi}_{A_1} \oplus \tilde{\Psi}_{P_{A_2}} : F^* \times \mathbb{C}^3 \rightarrow \mathbb{C}^5, \quad (f, x, y, \eta) \mapsto (f(x, y), f_x, f_y, f_{xx} + \eta f_{xy}, f_{xy} + \eta f_{yy}) \]
is transverse to zero at \((f, 0, 0, 0)\). The Jacobian matrix of this map at \((f, 0, 0, 0)\) is a \(5 \times (\delta_d + 4)\) matrix which has full rank if \(d \geq 2\). This is easy to see if the first five columns of the matrix are partial derivatives with respect to \(f_00, f_{10}, f_{01}, f_{20}\) and \(f_{11}\).

**Proof of Corollary 1.6.14:** This follows immediately from Lemma 1.6.1 and Proposition 1.6.13.

**Proof of Proposition 1.6.15:** Equation (1.6.10) follows from Lemma 1.4.5 and Corollary 1.4.7. To see this, we observe that Corollary 1.4.7 (for \(k = 2\)) gives us a necessary and sufficient condition for a curve \(\rho^{-1}(0)\) to have an \(A_2\)-node. In terms of bundle sections, this is equivalent to the statement that the space \(\mathcal{P}_{A_2}\) can be described as
\[ \mathcal{P}_{A_2} = \{ (\tilde{f}, l_p) \in \mathcal{D} \times \mathbb{P}T^2 : \Psi_{A_0}(\tilde{f}, l_p) = 0, \Psi_{A_1}(\tilde{f}, l_p) = 0, \Psi_{P_{A_2}}(\tilde{f}, l_p) = 0, \Psi_{P_{A_3}}(\tilde{f}, l_p) \neq 0, \Psi_{P_{D_4}}(\tilde{f}, l_p) \neq 0 \}. \]

The desired equation (1.6.10) now follows from Corollary 1.6.10 and 1.6.14.

We will now show transversality, having already proved it for the section \(\Psi_{\mathcal{P}_{A_2}}\) in the proof of Proposition 1.6.13. Let us start with \(\Psi_{\mathcal{P}_{A_3}}\). By Lemma 1.4.18, statement 2 we conclude that restricted to \(\Psi_{\mathcal{P}_{A_3}}^{-1}(0)\), the section \(\Psi_{\mathcal{P}_{A_3}}\) is well defined. As before, showing that the section \(\Psi_{\mathcal{P}_{A_3}}\) is transverse to the zero set is equivalent to showing that the map
\[ \tilde{\Psi}_{A_0} \oplus \tilde{\Psi}_{A_1} \oplus \tilde{\Psi}_{P_{A_2}} \oplus \tilde{\Psi}_{P_{A_3}} : F^* \times \mathbb{C}^3 \rightarrow \mathbb{C}^6, \quad (f, x, y, \eta) \mapsto (f(x, y), f_x, f_y, f_{xx} + \eta f_{xy}, f_{xy} + \eta f_{yy}, f_{xxy}) \]
is transverse to zero, where
\[ \tilde{x} := x + \eta y \quad \text{and} \quad f_{xxy} := (\partial_x + \eta \partial_y)^3 f(x, y). \]
The Jacobian matrix of this map at \((f, 0, 0, 0)\) is a \(6 \times (\delta_d + 4)\) matrix which has full rank if \(d \geq 3\). This follows by looking at the first six columns of the matrix which are partial derivatives with respect to \(f_{00}, f_{10}, f_{01}, f_{20}, f_{11}\) and \(f_{30}\). Next let us show the transversality of \(\Psi_{\mathcal{P}_{D_4}}\). Note that \(\Psi_{\mathcal{P}_{D_4}}\) is well defined restricted to \(\Psi_{\mathcal{P}_{A_2}}^{-1}(0)\) since \(\nabla^2 f|_p\) is well defined (cf. Lemma 1.4.18, statement 1). Showing that the section \(\Psi_{\mathcal{P}_{D_4}}\) is transverse to the zero is equivalent to showing that the map
\[ \tilde{\Psi}_{A_0} \oplus \tilde{\Psi}_{A_1} \oplus \tilde{\Psi}_{P_{A_2}} \oplus \tilde{\Psi}_{P_{D_4}} : F^* \times \mathbb{C}^3 \rightarrow \mathbb{C}^6, \quad (f, x, y, \eta) \mapsto (f(x, y), f_x, f_y, f_{xx} + \eta f_{xy}, f_{xy} + \eta f_{yy}, f_{yy}) \]
is transverse to zero at \((f, 0, 0, 0)\). The Jacobian matrix of this map at \((f, 0, 0, 0)\) is a \(6 \times (\delta_d + 4)\) matrix which has full rank if \(d \geq 2\); look at the first six columns which are partial derivatives with respect to \(f_{00}, f_{10}, f_{01}, f_{20}, f_{11}\) and \(f_{02}\).

**Proof of Corollary 1.6.16:** This follows immediately from Lemma 1.6.3 and Proposition 1.6.15.
Proof of Proposition 1.6.17: Equation (1.6.11) is the definition of \( \hat{D}_\theta \). Towards showing transversality, note that the transversality of the sections \( \Psi_{\tilde{A}_0} \) and \( \Psi_{\tilde{A}_1} \) is taken care of in the proof of Proposition 1.6.13. Moreover, proving the transversality of \( \Psi_{\tilde{D}_4} \) is identical to the proof of transversality of \( \psi_{D_4} \) in Proposition 1.6.6. We will now show the transversality of the section \( \Psi_{P_{A_3}} \) restricted to \( \tilde{D}_4(0) \). Let \((\tilde{f}, l_\tilde{p}) \in \tilde{D}_4(0)\). Then it is easy to see that \( \Psi_{P_{A_2}}(\tilde{f}, l_\tilde{p}) = 0 \). Hence, \( \Psi_{P_{A_3}} \) is well defined (cf. Lemma 1.4.18, statement 2). As before, showing that the section \( \Psi_{P_{A_3}} \) is transverse to the zero set is equivalent to showing that the map

\[
\tilde{\Psi}_{\tilde{A}_0} \oplus \tilde{\Psi}_{\tilde{A}_1} \oplus \tilde{\Psi}_{\tilde{P}_{A_2}} \oplus \tilde{\Psi}_{\tilde{P}_{A_3}} \oplus \tilde{\Psi}_{\tilde{P}_{D_4}} : \mathcal{F}^* \times \mathbb{C}^3 \rightarrow \mathbb{C}^7,
\]

is transverse to zero and \( \hat{x} = x + \eta y \). The Jacobian matrix of this map at \((\tilde{f}, 0, 0, 0)\) is a \( 7 \times (\delta_d + 4) \) matrix which has full rank if \( d \geq 3 \); choose the first seven columns as the partial derivatives with respect to \( f_0, f_{10}, f_{20}, f_{11}, f_{02} \) and \( f_{30} \).

Proof of Corollary 1.6.18: This follows immediately from Lemma 1.6.1 and Proposition 1.6.17.

Proof of Proposition 1.6.19: Proving (1.6.12) requires some care. First let us define

\[
\beta := f_{30}^2 f_{03}^3 - 6 f_{03} f_{12} f_{21} f_{30} + 4 f_{12}^2 f_{30} + 4 f_{03} f_{21}^3 - 3 f_{12}^2 f_{21}^2.
\]

This is the discriminant of the cubic term in the Taylor expansion of \( f \). Since \((\tilde{f}, l_\tilde{p}) \in \mathcal{P}D_4 \), we conclude using Corollary 1.4.10 that

\[
\beta \neq 0, \quad f_{30} = 0 \quad \implies \quad f_{21} \neq 0 \quad \text{and} \quad 3 f_{12}^2 - 4 f_{21} f_{03} \neq 0.
\]

Hence,

\[
\mathcal{P}D_4 = \{(\tilde{f}, l_\tilde{p}) \in \mathcal{D} \times \mathbb{P}T \mathbb{P}^2 : \Psi_{\tilde{A}_0}(\tilde{f}, l_\tilde{p}) = 0, \Psi_{\tilde{A}_1}(\tilde{f}, l_\tilde{p}) = 0, \Psi_{\tilde{P}_{A_4}}(\tilde{f}, l_\tilde{p}) = 0, \Psi_{\tilde{P}_{A_2}}(\tilde{f}, l_\tilde{p}) = 0, \Psi_{\tilde{P}_{A_3}}(\tilde{f}, l_\tilde{p}) = 0, \Psi_{\tilde{P}_{D_4}}(\tilde{f}, l_\tilde{p}) = 0, \Psi_{\tilde{P}_{D_4}'}(\tilde{f}, l_\tilde{p}) \neq 0, \Psi_{\tilde{P}_{D_4}''}(\tilde{f}, l_\tilde{p}) \neq 0 \}.
\]

(1.6.25)

It is evident via linear algebra that

\[
\Psi_{\tilde{P}_{A_4}}(\tilde{f}, l_\tilde{p}) = 0, \quad \Psi_{\tilde{P}_{D_4}}(\tilde{f}, l_\tilde{p}) \neq 0, \quad \Psi_{\tilde{P}_{D_4}'}(\tilde{f}, l_\tilde{p}) \neq 0, \quad \Psi_{\tilde{P}_{A_3}}(\tilde{f}, l_\tilde{p}) = 0
\]

\[(1.6.26)\]

Using equations (1.6.25) and (1.6.26), we get that

\[
\mathcal{P}D_4 = \{(\tilde{f}, l_\tilde{p}) \in \mathcal{D} \times \mathbb{P}T \mathbb{P}^2 : \Psi_{\tilde{A}_0}(\tilde{f}, l_\tilde{p}) = 0, \Psi_{\tilde{A}_1}(\tilde{f}, l_\tilde{p}) = 0, \Psi_{\tilde{A}_2}(\tilde{f}, l_\tilde{p}) = 0, \Psi_{\tilde{A}_3}(\tilde{f}, l_\tilde{p}) = 0, \Psi_{\tilde{P}_{D_4}}(\tilde{f}, l_\tilde{p}) = 0, \Psi_{\tilde{P}_{D_4}'}(\tilde{f}, l_\tilde{p}) \neq 0, \Psi_{\tilde{P}_{D_4}''}(\tilde{f}, l_\tilde{p}) \neq 0 \}.
\]

(1.6.27)
\[(f, x, y, \eta) \mapsto (f(x, y), f_x, f_y, f_{xx} + \eta f_{xy}, f_{xy} + \eta f_{yy}, f_{x\hat{x}}, f_{y\hat{y}})\]

is transverse to zero at \((f, 0, 0, 0)\), where \(\hat{x} = x + \eta y\). The Jacobian matrix of this map at \((f, 0, 0, 0)\) is a \(7 \times (\delta_d + 4)\) matrix which has full rank if \(d \geq 3\); take the first seven columns as the partial derivatives with respect to \(f_0, f_1, f_0, f_2, f_0, f_1, f_0, f_0, f_0\).

Next let us show that \(\Psi_D^{-1}\) is transverse. Note that restricted to \(\Psi_D^{-1}(0)\), the section \(\Psi_D^{-1}\) is well defined; when restricted to \(\Psi_D^{-1}(0)\), we infer that \(f(p) = 0\), \(\nabla f_x = 0\) and \(\nabla^2 f|_p = 0\). Hence, by Lemma 1.4.18, statement 1, the quantity \(\nabla^3 f|_p\) is well defined. Consequently, the section \(\Psi_D^{-1}\) is also well defined. Showing that the section \(\Psi_D^{-1}\) is transverse to the zero set is equivalent to showing that the map

\[\tilde{\Psi}_A_0 \oplus \tilde{\Psi}_A_1 \oplus \tilde{\Psi}_{P,A_2} \oplus \tilde{\Psi}_{P,A_3} \oplus \tilde{\Psi}_{P,D_4} \oplus \tilde{\Psi}_{P,D_5} : F^* \times \mathbb{C}^3 \longrightarrow \mathbb{C}^8,\]

\[(f, x, y, \eta) \mapsto (f(x, y), f_x, f_y, f_{xx} + \eta f_{xy}, f_{xy} + \eta f_{yy}, f_{x\hat{x}}, f_{y\hat{y}}, f_{x\hat{x}\hat{y}})\]

is transverse to zero at \((f, 0, 0, 0)\), where \(\hat{x} = x + \eta y\). The Jacobian matrix of this map at \((f, 0, 0, 0)\) is an \(8 \times (\delta_d + 4)\) matrix which has full rank if \(d \geq 3\); take the first eight columns as the partial derivatives with respect to \(f_0, f_1, f_0, f_2, f_0, f_1, f_0, f_0, f_0\) and \(f_21\).

Finally, let us show that \(\Psi_D^{-1}\) is transverse. Since \(\nabla^3 f|_p\) is well defined restricted to \(\Psi_D^{-1}(0)\), we conclude the section \(\Psi_D^{-1}\) is well defined when restricted to \(\Psi_D^{-1}(0)\). Showing that the section \(\Psi_D^{-1}\) is transverse to the zero set is equivalent to showing that the map

\[\tilde{\Psi}_A_0 \oplus \tilde{\Psi}_A_1 \oplus \tilde{\Psi}_{P,A_2} \oplus \tilde{\Psi}_{P,A_3} \oplus \tilde{\Psi}_{P,D_4} \oplus \tilde{\Psi}_{P,D_5} : F^* \times \mathbb{C}^3 \longrightarrow \mathbb{C}^8,\]

\[(f, x, y, \eta) \mapsto (f(x, y), f_x, f_y, f_{xx} + \eta f_{xy}, f_{xy} + \eta f_{yy}, f_{x\hat{x}}, f_{y\hat{y}}, f_{x\hat{x}\hat{y}}, f_{x\hat{y}y}, f_{x\hat{y}x}, f_{x\hat{x}\hat{y}})\]

is transverse to zero at \((f, 0, 0, 0)\), when \(f_21 \neq 0\) where \(\hat{x} = x + \eta y\). The Jacobian matrix of this map at \((f, 0, 0, 0)\) is an \(8 \times (\delta_d + 4)\) matrix which has full rank if \(d \geq 3\); take the first eight columns as the partial derivatives with respect to \(f_0, f_1, f_0, f_2, f_0, f_1, f_0, f_0, f_0\) and \(f_03\).

Proof of Corollary 1.6.20: This follows immediately from Lemma 1.6.3 and Proposition 1.6.19.9

Proof of Proposition 1.6.21: By Corollary 1.4.7 (for \(k = 3\)), we get that

\[P A_3 = \{ (\tilde{f}, l_{\tilde{p}}) \in D \times \mathbb{R}^P \sqcup \Psi A_0(\tilde{f}, l_{\tilde{p}}) = 0, \ \Psi A_1(\tilde{f}, l_{\tilde{p}}) = 0, \ \Psi P A_2(\tilde{f}, l_{\tilde{p}}) = 0, \ \Psi P A_3(\tilde{f}, l_{\tilde{p}}) = 0, \ \Psi P D_4(\tilde{f}, l_{\tilde{p}}) \neq 0 \}.\] 

Equation (1.6.13) now follows from (1.6.28) and Corollary 1.6.16, 1.6.14 and 1.6.10.

Towards showing transversality of the bundle sections, note that the sections \(\Psi PA_3\) and \(\Psi PD_4\) are already transverse (cf. Proposition 1.6.15 and 1.6.19). We now prove transversality of \(\Psi PA_4\). By Lemma 1.4.18, statement 3 we conclude that \(\Psi PA_4\) is well defined. Showing that this section is transverse to the zero set is equivalent to showing that the map

\[\tilde{\Psi}_A_0 \oplus \tilde{\Psi}_A_1 \oplus \tilde{\Psi}_{P,A_2} \oplus \tilde{\Psi}_{P,A_3} \oplus \tilde{\Psi}_{P,A_4} : F^* \times \mathbb{C}^3 \longrightarrow \mathbb{C}^7,\]

\[(f, x, y, \eta) \mapsto (f(x, y), f_x, f_y, f_{xx} + \eta f_{xy}, f_{xy} + \eta f_{yy}, f_{x\hat{x}}, f_{y\hat{y}} A_4 f_{x\hat{y}})\] (1.6.29)

9Take \(M\) to be \(\Psi_D^{-1}(0)\) in Lemma 1.6.3.
is transverse to zero at \((f,0,0,0)\), where \(\hat{x} = x + \eta y\). From (1.4.4) it follows that

\[
f_{yy} A_4 f(\hat{x},y) = f_{yy} f_{\hat{x}\hat{x}\hat{x}} - 3 f_{\hat{x}y}.
\]

Hence, the Jacobian matrix of this map at \((f,0,0,0)\) is a \(7 \times (\delta_d + 4)\) matrix which has full rank if \(d \geq 4\) and \(f_{02} \neq 0\); take the first seven columns as the partial derivatives with respect to \(f_{00}, f_{10}, f_{01}, f_{20}, f_{11}, f_{30}\) and \(f_{40}\). Notice that the condition \(\Psi_{D_0}(\hat{f},l_\hat{p}) \neq 0\) (which is equivalent to \(f_{02} \neq 0\)) is necessary to conclude that the Jacobian matrix has full rank.

**Proof of Corollary 1.6.22**: This follows immediately from Lemma 1.6.3 and Proposition 1.6.21.

**Proof of Corollary 1.6.23**: We observe that \(A_3 = \pi(PA_3)\), where \(\pi : D \times \mathbb{P}T\mathbb{P}^2 \rightarrow D \times \mathbb{P}^2\) is the projection map. Let us continue with the setup of Proposition 1.6.21 and consider (1.6.29). It suffices to show that the zero set of the map

\[
F^* \times \mathbb{C}^2 \rightarrow \mathbb{C}^5, \quad (f,x,y) \mapsto \left(f(x,y), f_x, f_y, f_x f_y f_{xy} - f_{x^2 y} + (\partial_x - \frac{f_y}{f_{yy}} \partial_y)^3 f(x,y)\right)
\]

is smooth submanifold of \(F^* \times \mathbb{C}^2\) at \((f,0,0)\). The Jacobian of this map at \((f,0,0)\) is a \(5 \times (\delta_d + 3)\) matrix which has full rank if \(d \geq 3\); take the first five columns as the partial derivatives with respect to \(f_{00}, f_{10}, f_{01}, f_{20}\) and \(f_{30}\).

**Proof of Proposition 1.6.24**: By Corollary 1.4.7, we get that

\[
PA_k = \{ (\hat{f},l_\hat{p}) \in D \times \mathbb{P}T\mathbb{P}^2 : \Psi_{A_0}(\hat{f},l_\hat{p}) = 0, \Psi_{A_1}(\hat{f},l_\hat{p}) = 0, \Psi_{PA_2}(\hat{f},l_\hat{p}) = 0, \ldots, \Psi_{PA_k}(\hat{f},l_\hat{p}) = 0, \Psi_{PA_{k+1}}(\hat{f},l_\hat{p}) \neq 0, \Psi_{PA_\iota}(\hat{f},l_\hat{p}) \neq 0\},
\]

Equation (1.6.14) now follows from (1.6.30) and Corollary 1.6.16, 1.6.14 and 1.6.10.

Towards proving transversality of the bundle section \(\Psi_{PA_i}\), note that by Lemma 1.4.18 and statement 3 this section is well defined. Showing that the section \(\Psi_{PA_i}\) is transverse to the zero set is equivalent to showing that the map

\[
\tilde{\Psi}_{A_0} \oplus \tilde{\Psi}_{A_1} \oplus \tilde{\Psi}_{PA_2} \oplus \tilde{\Psi}_{PA_3} \oplus \ldots \tilde{\Psi}_{PA_\iota} : F^* \times \mathbb{C}^3 \rightarrow \mathbb{C}^{i+3},
\]

\[
(f,x,y,\eta) \mapsto \left(f, f_x, f_y, f_{xx} + \eta f_{xy}, f_{xy} + \eta f_{yy}, f_{\hat{x}\hat{x}\hat{x}} + f_{\hat{x}y} A_4 f(\hat{x},y) + f_{yy} A_5 f(\hat{x},y), \ldots, f_{yy} A_k f(\hat{x},y)\right)
\]

is transverse to zero at \((f,0,0,0)\), where \(\hat{x} = x + \eta y\). Recall that \(A_i\) are defined in (1.4.3) (implicitly) and in (1.4.4) (explicitly till \(i = 7\)). Hence, the Jacobian matrix of this map at \((f,0,0,0)\) is an \((i+3) \times (\delta_d + 4)\) matrix which has full rank if \(d \geq i\) and \(f_{02} \neq 0\); take the first \((i+3)\) columns of the matrix as partial derivatives with respect to \(f_{00}, f_{10}, f_{01}, f_{20}, f_{11}, f_{30}, f_{40}\) and \(f_{30}\). Notice that the condition \(\Psi_{PA_\iota}(\hat{f},l_\hat{p}) \neq 0\) (which is equivalent to \(f_{02} \neq 0\)) is necessary to conclude that the Jacobian matrix has full rank.

**Proof of Corollary 1.6.25**: This is similar to the proof of Corollary 1.6.23. We observe that \(A_k = \pi(PA_k)\), where \(\pi : D \times \mathbb{P}T\mathbb{P}^2 \rightarrow D \times \mathbb{P}^2\) is the projection map. It suffices to show that the zero set of the map \(F^* \times \mathbb{C}^2 \rightarrow \mathbb{C}^{k+2}\) given by

\[
(f,x,y) \mapsto \left(f(\hat{x},y), f_x, f_y, f_{xx}f_{yy} - f_{x^2 y}, f_{\hat{x}\hat{x}\hat{x}}, f_{yy} A_4 f(\hat{x},y) + f_{yy} A_5 f(\hat{x},y), \ldots, f_{yy} A_k f(\hat{x},y)\right)
\]
where \( \hat{x} = x - \frac{f_y}{f_x} y \), is smooth submanifold of \( \mathcal{F}^* \times \mathbb{C}^{k+2} \) at \((f,0,0)\). The Jacobian of this map at \((f,0,0)\) is a \((k + 2) \times (\delta_d + 3)\) matrix which has full rank if \( d \geq k \); take the first \((k + 2)\) columns as the partial derivatives with respect to \(f_0, f_1, f_2, f_30, \ldots, f_k0\). Notice that we require \( f_02 \neq 0 \) to conclude that the matrix has full rank.

**Proof of Proposition 1.6.26:** Proving (1.6.15) requires some care. Unravelling the definition of \( \mathcal{PD}_5 \) and using Corollary 1.4.12 for \( k = 5 \) we see that

\[
\mathcal{PD}_5 := \{(f, l_p) \in \mathcal{D} \times \mathbb{P}^2 : \Psi_{\hat{A}_0}(\hat{f}, l_p) = 0, \quad \Psi_{\hat{A}_1}(\hat{f}, l_p) = 0, \quad \Psi_{\hat{D}_4}(\hat{f}, l_p) = 0, \quad \Psi_{\hat{D}_5}(\hat{f}, l_p) = 0, \quad \Psi_{\mathcal{PD}_0}(\hat{f}, l_p) = 0, \quad \Psi_{\mathcal{PE}_0}(\hat{f}, l_p) = 0\}.
\]

(1.6.32)

Standard linear algebra implies that

\[
\begin{align*}
\Psi_{\hat{D}_4}(\hat{f}, l_p) &= 0, \quad \Psi_{\hat{D}_5}(\hat{f}, l_p) = 0 \\
\iff \quad \Psi_{\mathcal{PD}_5}(\hat{f}, l_p) &= 0
\end{align*}
\]

Hence, using (1.6.32) and (1.6.33), we get that

\[
\mathcal{PD}_5 = \{(f, l_p) \in \mathcal{D} \times \mathbb{P}^2 : \Psi_{\hat{A}_0}(\hat{f}, l_p) = 0, \Psi_{\hat{A}_1}(\hat{f}, l_p) = 0, \Psi_{\mathcal{PD}_0}(\hat{f}, l_p) = 0, \Psi_{\mathcal{PE}_0}(\hat{f}, l_p) = 0\}.
\]

(1.6.34)

Equation (1.6.15) now follows from (1.6.34), Corollary 1.6.20, 1.6.16, 1.6.14 and 1.6.10. We have already proved the transversality of the section \( \Psi_{\mathcal{PD}_0} \) in Proposition 1.6.19. We will now show the transversality of the sections \( \Psi_{\mathcal{PD}_5} \) and \( \Psi_{\mathcal{PE}_0} \). By Lemma 1.4.18, statement 4 and 6, these two sections are well defined. Showing that the section \( \Psi_{\mathcal{PD}_0} \) is transverse to the zero set is equivalent to showing that the map

\[
\psi_{\hat{A}_0} + \psi_{\hat{A}_1} + \psi_{\mathcal{PD}_5} + \psi_{\mathcal{PE}_0} : \mathcal{F}^* \times \mathbb{C}^3 \longrightarrow \mathbb{C}^9,
\]

\[
(f, x, y, \eta) \mapsto (f(x, y), f_x, f_y, f_{xx} + \eta f_{xy}, f_{xy} + \eta f_{yy}, f_{\hat{x}x}, f_{yy}, f_{\hat{x}y}, f_{\hat{x}\hat{y}})
\]

is transverse to zero at \((f,0,0,0)\), where \( \hat{x} = x + \eta y \). The Jacobian matrix of this map at \((f,0,0,0)\) is a \(9 \times (\delta_d + 4)\) matrix which has full rank if \( d \geq 4\); take the first 9 columns as the partial derivatives with respect to \(f_0, f_1, f_2, f_30, f_31, f_32, f_30, f_30, f_22, f_40\).

Showing that the section \( \Psi_{\mathcal{PD}_5} \) is transverse to the zero set is equivalent to showing that the map

\[
\psi_{\hat{A}_0} + \psi_{\hat{A}_1} + \psi_{\mathcal{PD}_5} + \psi_{\mathcal{PE}_0} : \mathcal{F}^* \times \mathbb{C}^3 \longrightarrow \mathbb{C}^9,
\]

\[
(f, x, y, \eta) \mapsto (f(x, y), f_x, f_y, f_{xx} + \eta f_{xy}, f_{xy} + \eta f_{yy}, f_{\hat{x}x}, f_{yy}, f_{\hat{x}y}, f_{\hat{x}\hat{y}})
\]

is transverse to zero at \((f,0,0,0)\), where \( \hat{x} = x + \eta y \). The Jacobian matrix of this map at \((f,0,0,0)\) is a \(9 \times (\delta_d + 4)\) matrix which has full rank if \( d \geq 4\); take the first 9 columns as the partial derivatives with respect to \(f_0, f_1, f_2, f_30, f_30, f_30, f_22, f_22, f_{i2} \). 

**Proof of Corollary 1.6.27:** This follows immediately from Lemma 1.6.3 and Proposition 1.6.26. 


Proof of Proposition 1.6.28: Proving (1.6.16) requires some care. Since \((\tilde{f}, \tilde{p}, l_p) \in \Delta PD_5^\vee\), we claim
\[
f_{21} \neq 0, \quad \beta_1 := \frac{(f_{12}\partial_x - 2f_{21}\partial_y)^3 f}{2f_{21}^2} = 3f_{12}^2 - 4f_{21}f_{03} = 0 \quad \text{and} \quad \beta_2 := (f_{12}\partial_x - 2f_{21}\partial_y)^4 f = f_{12}f_{40} - 8f_{12}^3f_{21} + 24f_{12}^2f_{21}f_{22} - 32f_{12}f_{21}f_{13} + 16f_{21}f_{04} \neq 0.
\] (1.6.35)

It is easy to see that (1.6.35) and (1.6.36) imply (1.6.16). Let us now justify (1.6.35) and (1.6.36). Since \((\tilde{f}, \tilde{p}) \in D_5\) there exists a non zero vector \(u = m_1v + m_2w\) such that
\[
\nabla^3f_\tilde{p}(u, u, v) = m_1^2f_{30} + 2m_1m_2f_{21} + m_2^2f_{12} = 0, \quad (1.6.37)
\]
and
\[
\nabla^3f_\tilde{p}(u, u, w) = m_1^2f_{21} + 2m_1m_2f_{12} + m_2^2f_{03} = 0, \quad (1.6.38)
\]
\[
\nabla^4f_\tilde{p}(u, u, u) \neq 0. \quad (1.6.39)
\]

Since \((\tilde{f}, l_p) \in PD_5^\vee\), we conclude by definition that
\[
f_{30} = 0, \quad f_{21} \neq 0, \quad m_2 \neq 0. \quad (1.6.40)
\]

(If \(m_2 = 0\) then \(f_{21}\) would be zero). Equations (1.6.40) and (1.6.37) now imply that
\[
\frac{m_1}{m_2} = \frac{f_{12}}{2f_{21}}. \quad (1.6.41)
\]

Equation (1.6.41) and (1.6.38) implies (1.6.35). Finally, (1.6.39) implies (1.6.36).

We have already shown \(\tilde{\Psi}_{PD_5}\) and \(\Psi_{PD_5}^\vee\) are transverse to the zero set in Proposition 1.6.19. We will now show the transversality of the section \(\Psi_{PD_6}^\vee\). The reason why \(\Psi_{PD_6}^\vee\) is well defined is similar to why \(\Psi_{PD_5}\) is well defined. Showing that the section \(\Psi_{PD_6}^\vee\) is transverse to the zero set is equivalent to showing that the map
\[
\tilde{\Psi}_A_0 \oplus \tilde{\Psi}_A_1 \oplus \tilde{\Psi}_A_2 \oplus \tilde{\Psi}_A_3 \oplus \tilde{\Psi}_{PD_4} \oplus \tilde{\Psi}_{PD_5}^\vee \oplus \tilde{\Psi}_{PD_6}^\vee : \mathbb{F}^* \times \mathbb{C}^3 \rightarrow \mathbb{C}^9,
\]
\[
(f, x, y, \eta) \mapsto (f(x, y), f_x, f_y, f_{xx} + \eta f_{xy}, f_{xy} + \eta f_{yy}, f_{xxy}, f_{yy}, \beta_1(\hat{x}), \beta_2(\hat{x}))
\]
is transverse to zero at \((f, 0, 0, 0)\) when \(f_{21} \neq 0\), where \(\hat{x} = x + \eta y\) and \(\beta_i(\hat{x}) = \beta_i\), but with partial with respect to \(x\) replaced by partial with respect to \(\hat{x}\). The Jacobian matrix of this map at \((f, 0, 0, 0)\) is a \(9 \times (\delta_d + 4)\) matrix which has full rank if \(d \geq 4\); take the first 9 columns as the partial derivatives with respect to \(f_{00}, f_{10}, f_{01}, f_{20}, f_{11}, f_{30}, f_{02}, f_{03}\) and \(f_{04}\). Note that \(f_{21} \neq 0\) is crucial here. □

Proof of Corollary 1.6.29: This follows immediately from Lemma 1.6.3 and Proposition 1.6.28. □

Proof of Corollary 1.6.30: This basically follows from the setup of Proposition 1.6.26. The proof is similar to the proof of Corollary 1.6.23 and 1.6.25. □

Proof of Proposition 1.6.31: Proving (1.6.17) requires some care. Unravelling the definition of \(P\mathcal{E}_0\) and using Corollary 1.4.15 we conclude that
\[
\mathcal{P}\mathcal{E}_0 = \{(\tilde{f}, l_p) \in D \times PD^2 : \Psi_{A_0}^\vee(\tilde{f}, l_p) = 0, \quad \Psi_{A_1}^\vee(\tilde{f}, l_p) = 0, \quad \Psi_{A_2}^\vee(\tilde{f}, l_p) = 0, \quad \Psi_{A_3}^\vee(\tilde{f}, l_p) = 0, \quad \Psi_{PD_4}(\tilde{f}, l_p) = 0, \quad \Psi_{PD_5}(\tilde{f}, l_p) = 0, \quad \Psi_{PD_6}(\tilde{f}, l_p) = 0, \quad \Psi_{PD_7}(\tilde{f}, l_p) \neq 0, \quad \Psi_{PD_8}(\tilde{f}, l_p) \neq 0, \quad \Psi_{PD_9}(\tilde{f}, l_p) \neq 0\}.
\] (1.6.42)
Using (1.6.42) and (1.6.33) we get that

\[ \mathcal{P}E_6 = \{ (f, l_p) \in D \times \mathbb{R}^2 : \Psi_{A_0}(f, l_p) = 0, \Psi_{A_1}(f, l_p) = 0, \Psi_{P_{A_2}}(f, l_p) = 0, \Psi_{P_{A_3}}(f, l_p) = 0, \Psi_{P_{D_4}}(f, l_p) = 0, \Psi_{P_{E_6}}(f, l_p) = 0, \Psi_{P_{E_7}}(f, l_p) = 0, \Psi_{P_{X_6}}(f, l_p) \neq 0 \}. \]  

(1.6.43)

Equation (1.6.17) now follows from (1.6.43) and Corollary 1.6.27, 1.6.20, 1.6.16, 1.6.14 and 1.6.10.

We have already proved the transversality of the sections \( \Psi_{P_{E_6}} \) in Proposition 1.6.26. We will now show the transversality of the sections \( \Psi_{P_{E_7}} \) and \( \Psi_{P_{X_6}} \). Let us start with \( \Psi_{P_{E_7}} \). By Lemma 1.4.18, statement 7, the section is well defined. Showing that the section \( \Psi_{P_{E_7}} \) is transverse to the zero set is equivalent to showing that the map

\[ \Psi_{A_0} \oplus \Psi_{A_1} \oplus \Psi_{P_{A_2}} \oplus \Psi_{P_{A_3}} \oplus \Psi_{P_{D_4}} \oplus \Psi_{P_{D_5}} \oplus \Psi_{P_{E_6}} \oplus \Psi_{P_{E_7}} : \mathcal{F}^* \times \mathbb{C}^3 \to \mathbb{C}^{10}, \]

\( (f, x, y, \eta) \mapsto (f(x, y), f_x, f_y, f_{xx} + \eta f_{xy}, f_{xy} + \eta f_{yy}, f_{x^2x}, f_{xyy}, f_{x^2y}, f_{x^2y}x) \)

is transverse to zero at \((f, 0, 0, 0)\), where \( \hat{x} = x + \eta y \). The Jacobian of this map at \((f, 0, 0, 0)\) is a \(10 \times (\delta_d + 3)\) matrix which has full rank if \(d \geq 4\); take the first 10 columns as the partial derivatives with respect to \(f_00, f_{10}, f_{01}, f_{20}, f_{11}, f_{30}, f_{02}, f_{21}, f_{12} \) and \(f_{03} \).

Observe that restricted to \( \Psi_{P_{E_6}}^{-1}(0) \), \( \nabla^2 f|_{\eta} = 0 \), whence \( \Psi_{P_{X_6}} \) is well defined by Lemma 1.4.18, statement 1. Showing that this section is transverse to the zero set is equivalent to showing that the map

\[ \Psi_{A_0} \oplus \Psi_{A_1} \oplus \Psi_{P_{A_2}} \oplus \Psi_{P_{A_3}} \oplus \Psi_{P_{D_4}} \oplus \Psi_{P_{D_5}} \oplus \Psi_{P_{E_6}} \oplus \Psi_{P_{X_6}} : \mathcal{F}^* \times \mathbb{C}^3 \to \mathbb{C}^{10}, \]

\( (f, x, y, \eta) \mapsto (f(x, y), f_x, f_y, f_{xx} + \eta f_{xy}, f_{xy} + \eta f_{yy}, f_{x^2x}, f_{xyy}, f_{x^2y}, f_{x^2y}x) \)

is transverse to zero at \((f, 0, 0, 0)\), where \( \hat{x} = x + \eta y \). The Jacobian of this matrix at \((f, 0, 0, 0)\) is a \(10 \times (\delta_d + 4)\) matrix which has full rank if \(d \geq 3\); take the first 10 columns as the partial derivatives with respect to \(f_00, f_{10}, f_{01}, f_{20}, f_{11}, f_{30}, f_{02}, f_{21}, f_{12} \) and \(f_{03} \).

\[ \square \]

**Proof of Corollary 1.6.32:** This follows immediately from Lemma 1.6.3 and Proposition 1.6.31.

\[ \square \]

**Proof of Corollary 1.6.33:** This is identical to the setup of Proposition 1.6.31. The proof is similar to the proof of Corollary 1.6.23 and 1.6.25.

\[ \square \]

**Proof of Proposition 1.6.34:** Proving (1.6.18) requires some care. We begin by observing that Corollary 1.4.12 (for \(k = 6\)) implies that

\[ \mathcal{P}D_6 = \{ (f, l_p) \in D \times \mathbb{R}^2 : \Psi_{A_0}(f, l_p) = 0, \Psi_{A_1}(f, l_p) = 0, \Psi_{D_4}(f, l_p) = 0, \Psi_{D_6}(f, l_p) = 0, \Psi_{E_6}(f, l_p) = 0, \Psi_{P_{D_6}}(f, l_p) = 0, \Psi_{P_{E_7}}(f, l_p) = 0, \Psi_{P_{X_6}}(f, l_p) \neq 0 \}. \]  

(1.6.44)

Using (1.6.44) and (1.6.33) we conclude that

\[ \mathcal{P}D_6 = \{ (f, l_p) \in D \times \mathbb{R}^2 : \Psi_{A_0}(f, l_p) = 0, \Psi_{A_1}(f, l_p) = 0, \Psi_{P_{A_2}}(f, l_p) = 0, \Psi_{P_{A_3}}(f, l_p) = 0, \Psi_{P_{D_4}}(f, l_p) = 0, \Psi_{P_{E_6}}(f, l_p) = 0, \Psi_{P_{E_7}}(f, l_p) = 0, \Psi_{P_{X_6}}(f, l_p) \neq 0 \}. \]  

(1.6.45)

Equation (1.6.18) now follows from (1.6.45) and Corollary 1.6.27, 1.6.20, 1.6.16, 1.6.14 and 1.6.10.

Towards proving transversality, note that we have already shown the transversality of \( \Psi_{P_{D_6}} \) in...
Proposition 1.6.26. We now prove transversality of $\Psi_{PE_6}$ and $\Psi_{PD_7}$. Let us start with $\Psi_{PE_6}$. Note that restricted to $\Psi^{-1}_{PD_6}(0)$, the quantity $\nabla^2 f|_p$ vanishes, whence the section $\Psi_{PE_6}$ is well defined (cf. Lemma 1.4.18, statement 1). Showing that this section is transverse to the zero set is equivalent to showing that the map

$$
\bar{\Psi}_{A_0} \oplus \bar{\Psi}_{A_1} \oplus \bar{\Psi}_{P_{A_2}} \oplus \bar{\Psi}_{P_{A_3}} \oplus \bar{\Psi}_{P_{D_4}} \oplus \bar{\Psi}_{PD_5} \oplus \bar{\Psi}_{PD_6} \oplus \bar{\Psi}_{PE_6} : \mathcal{F}^* \times \mathbb{C}^3 \longrightarrow \mathbb{C}^{10},
$$

$$(f, x, y, \eta) \mapsto (f(x, y), f_x, f_y, f_{xx} + \eta f_{xy}, f_{xy} + \eta f_{yy}, f_{xxy}, f_{xyy}, f_{xxx}, f_{xxyy})$$

is transverse to zero at $(f, 0, 0, 0)$, where $x = x + \eta y$. The Jacobian matrix of this map at $(f, 0, 0, 0)$ is a $10 \times (\delta_d + 4)$ matrix which has full rank if $d \geq 4$ and $f_{02} \neq 0$. This follows by taking the first ten columns to be partial derivatives with respect to $f_{00}$, $f_{10}$, $f_{01}$, $f_{20}$, $f_{11}$, $f_{30}$, $f_{02}$, $f_{21}$, $f_{40}$ and $f_{12}$.

Observe that $\Psi_{PD_7}$ is well defined by Lemma 1.4.18, statement 5. Showing that this section is transverse to the zero set is equivalent to showing that the map

$$
\bar{\Psi}_{A_0} \oplus \bar{\Psi}_{A_1} \oplus \bar{\Psi}_{P_{A_2}} \oplus \bar{\Psi}_{P_{A_3}} \oplus \bar{\Psi}_{P_{D_4}} \oplus \bar{\Psi}_{PD_5} \oplus \bar{\Psi}_{PD_6} \oplus \bar{\Psi}_{PD_7} : \mathcal{F}^* \times \mathbb{C}^3 \longrightarrow \mathbb{C}^{10},
$$

$$(f, x, y, \eta) \mapsto (f(x, y), f_x, f_y, f_{xx} + \eta f_{xy}, f_{xy} + \eta f_{yy}, f_{xxy}, f_{xyy}, f_{xxx}, f_{xxyy}, f_{xyyD_7^{(\hat{\eta},y)}})$$

is transverse to zero at $(f, 0, 0, 0)$, where $\hat{x} = x + \eta y$. From (1.3.10) we see that

$$
f_{\hat{x}yyD_7^{(\hat{x},y)}} = f_{\hat{x}yyf_{\hat{x}xyy}} - \frac{5f_{\hat{x}x^2y}}{3}.
$$

Hence, the Jacobian matrix of this map at $(f, 0, 0, 0)$ is a $10 \times (\delta_d + 4)$ matrix which has full rank if $d \geq 5$ and $f_{12} \neq 0$. This is evident by taking the first ten columns to be partial derivatives with respect to $f_{00}$, $f_{10}$, $f_{01}$, $f_{20}$, $f_{11}$, $f_{30}$, $f_{02}$, $f_{21}$, $f_{40}$ and $f_{50}$. Notice that the condition $f_{12} \neq 0$ is necessary to conclude that the matrix has full rank.

\textbf{Proof of Corollary 1.6.35:} This follows immediately from Lemma 1.6.3 and Proposition 1.6.34.

\textbf{Proof of Corollary 1.6.36:} This basically follows from the setup of Proposition 1.6.34. The proof is similar to the proof of Corollary 1.6.23 and 1.6.25.

\textbf{Proof of Proposition 1.6.37:} Proving (1.6.20) requires some care. Notice that Corollary 1.4.12 implies that

$$
P_{D_k} = \{ (\bar{f}, l_p) \in D \times \mathbb{P}T\mathbb{P}^2 : \Psi_{A_0}(\bar{f}, l_p) = 0, \Psi_{A_1}(\bar{f}, l_p) = 0, \Psi_{D_4}(\bar{f}, l_p) = 0, \Psi_{PD_5}(\bar{f}, l_p) = 0, \Psi_{PD_6}(\bar{f}, l_p) = 0, \Psi_{PD_7}(\bar{f}, l_p) = 0, \Psi_{PD_8}(\bar{f}, l_p) = 0, \Psi_{PD_9}(\bar{f}, l_p) = 0, \Psi_{PD_{10}}(\bar{f}, l_p) = 0, \Psi_{PD_{11}}(\bar{f}, l_p) = 0 \}.
$$

Equation (1.6.20) now follows from (1.6.46), (1.6.33) and Corollary 1.6.35, 1.6.27, 1.6.20, 1.6.16, 1.6.14 and 1.6.10.

We now prove transversality of the bundle sections $\Psi_{PD_7}$. By Lemma 1.4.18, statement 5 these sections are well defined. Showing that these sections are transverse to the zero set is equivalent to showing that the map

$$
\bar{\Psi}_{A_0} \oplus \bar{\Psi}_{A_1} \oplus \bar{\Psi}_{P_{A_2}} \oplus \bar{\Psi}_{P_{A_3}} \oplus \bar{\Psi}_{P_{D_4}} \oplus \bar{\Psi}_{PD_5} \oplus \bar{\Psi}_{PD_6} \oplus \ldots \bar{\Psi}_{PD_7} : \mathcal{F}^* \times \mathbb{C}^3 \longrightarrow \mathbb{C}^{i+3},
$$

$$(f, x, y, \eta) \mapsto \left( f(x, y), f_x, f_y, f_{xx} + \eta f_{xy}, f_{xy} + \eta f_{yy}, f_{xxy}, f_{xyy}, f_{xxx}, f_{xxyy}, D_7^{(\hat{x},y)}, \ldots, f_{12}^{\hat{x}yyD_7^{(\hat{x},y)}} \right)
$$

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is transverse to zero at \((f,0,0,0)\), where \(\hat{x} = x + \eta y\). Recall that \(\mathcal{D}_i^j\) is defined in (1.4.10) implicitly and in (1.3.10) explicitly till \(i = 8\). The Jacobian matrix of this map at \((f,0,0,0)\) is an \((i+3) \times (\delta_d+4)\) matrix which has full rank if \(d \geq i-2\) and \(f_{12} \neq 0\). This follows by choosing the first \((i+3)\) columns to be the partial derivatives with respect to \(f_{00}, f_{10}, f_{01}, f_{20}, f_{11}, f_{30}, f_{02}, f_{21}, f_{40}, f_{50}, \ldots\) and \(f_{1-2,0}\). Notice that the condition \(\Psi_{\mathcal{C}_6}(\hat{f}, l_{\hat{p}}) \neq 0\) (which is equivalent to \(f_{12} \neq 0\)) is necessary to conclude that the Jacobian matrix has full rank.

**Proof of Corollary 1.6.38**: This basically follows from the setup of Proposition 1.6.37. The proof is similar to the proof of Corollary 1.6.23 and 1.6.25.

**Proof of Proposition 1.6.39**: Proving (1.6.21) requires some care. Unravelling the definition of \(\mathcal{P}E_7\) in conjunction with Proposition 1.4.17 we gather

\[
\mathcal{P}E_7 = \{(\hat{f}, l_{\hat{p}}) \in \mathcal{D} \times \mathbb{P}^2 : \Psi_{\bar{A}_0}(\hat{f}, l_{\hat{p}}) = 0, \ \Psi_{\bar{A}_1}(\hat{f}, l_{\hat{p}}) = 0, \ \Psi_{\mathcal{D}_3}(\hat{f}, l_{\hat{p}}) = 0, \ \Psi_{\mathcal{P}D_5}(\hat{f}, l_{\hat{p}}) = 0, \ \Psi_{\mathcal{P}E_6}(\hat{f}, l_{\hat{p}}) = 0, \ \Psi_{\mathcal{P}E_7}(\hat{f}, l_{\hat{p}}) = 0, \ \Psi_{\mathcal{P}X_8}(\hat{f}, l_{\hat{p}}) \neq 0\}.
\]

Equation (1.6.21) now follows from equation (1.6.48) and Corollary 1.6.32, 1.6.27, 1.6.20, 1.6.16, 1.6.14 and 1.6.10.

We have already proved transversality of \(\Psi_{\mathcal{P}E_7}\) in Proposition 1.6.31. We now prove transversality of \(\Psi_{\mathcal{P}E_6}\) and \(\Psi_{\mathcal{P}X_8}\). Let's us start with \(\Psi_{\mathcal{P}E_6}\). By Lemma 1.4.18, statement 8 the section is well defined. Showing that this section is transverse to the zero set, is equivalent to showing that the map

\[
\Psi_{\bar{A}_0} \oplus \Psi_{\bar{A}_1} \oplus \Psi_{\mathcal{P}A_4} \oplus \Psi_{\mathcal{P}A_5} \oplus \Psi_{\mathcal{P}D_4} \oplus \Psi_{\mathcal{P}D_6} \oplus \Psi_{\mathcal{P}E_6} \oplus \Psi_{\mathcal{P}E_7} \oplus \Psi_{\mathcal{P}X_8} : \mathcal{F}^* \times \mathbb{C}^3 \rightarrow \mathbb{C}^{11},
\]

\[(f, x, y, \eta) \mapsto (f(x, y), f_x, f_y, \eta f_{xy}, f_{xx} + \eta f_{xy}, f_{xy} + \eta f_{yy}, f_{xxy}, f_{xyy}, f_{xxyy}, f_{xxyy}, f_{xxyy})\]

is transverse to zero at \((f,0,0,0)\), where \(\hat{x} = x + \eta y\). The Jacobian matrix of this map at \((f,0,0,0)\) is an \(11 \times (\delta_d+4)\) matrix which has full rank if \(d \geq 4\) and \(f_{12} \neq 0\). This follows by taking the first eleven columns as the partial derivatives with respect to \(f_{00}, f_{10}, f_{01}, f_{20}, f_{11}, f_{30}, f_{02}, f_{21}, f_{40}, f_{50}\) and \(f_{31}\).

To show that \(\Psi_{\mathcal{P}X_8}\) is transverse to the zero set, we observe that restricted to \(\Psi_{\mathcal{P}E_7}^{-1}(0)\), the quantity \(\nabla^3 |\mathcal{P}|\) is identically zero (via linear algebra). Hence, the section is well defined by Lemma 1.4.18, statement 1. Showing that this section is transverse to the zero set is equivalent to showing that the map

\[
\Psi_{\bar{A}_0} \oplus \Psi_{\bar{A}_1} \oplus \Psi_{\mathcal{P}A_4} \oplus \Psi_{\mathcal{P}A_5} \oplus \Psi_{\mathcal{P}D_4} \oplus \Psi_{\mathcal{P}D_6} \oplus \Psi_{\mathcal{P}E_6} \oplus \Psi_{\mathcal{P}E_7} \oplus \Psi_{\mathcal{P}X_8} : \mathcal{F}^* \times \mathbb{C}^3 \rightarrow \mathbb{C}^{11},
\]

\[(f, x, y, \eta) \mapsto (f(x, y), f_x, f_y, f_{xx} + \eta f_{xy}, f_{xy} + \eta f_{yy}, f_{xxy}, f_{xyy}, f_{xxyy}, f_{xxyy}, f_{xxyy})\]

is transverse to zero at \((f,0,0,0)\), where \(\hat{x} = x + \eta y\). The Jacobian matrix of this map at \((f,0,0,0)\) is an \(11 \times (\delta_d+4)\) matrix which has full rank if \(d \geq 4\) and \(f_{12} \neq 0\); take the first eleven columns as the partial derivatives with respect to \(f_{00}, f_{10}, f_{01}, f_{20}, f_{11}, f_{30}, f_{02}, f_{21}, f_{40}, f_{50}\) and \(f_{31}\). \(\square\)
Proof of Corollary 1.6.40: This follows immediately from Lemma 1.6.3 and Proposition 1.6.39. □

Proof of Corollary 1.6.41: This basically follows from the setup of Proposition 1.6.39. The proof is similar to the proof of Corollary 1.6.23 and 1.6.25. □

Proof of Proposition 1.6.42: Observe that $\Psi_{\mathcal{D}_0} = \Psi_{\mathcal{E}_7}$. Equation (1.6.22) follows from Proposition 1.6.39, Corollary 1.6.35 and 1.6.32. Transversality of $\Psi_{\mathcal{E}_7}$ has been proven in Proposition 1.6.34 (there it was denoted as $\Psi_{\mathcal{D}_0}$). Proving that the sections $\Psi_{\mathcal{E}_8}$ and $\Psi_{\mathcal{E}_9}$ are transverse to the zero set is almost identical to the proof in Proposition 1.6.39. □

Proof of Corollary 1.6.43: This follows immediately from Lemma 1.6.3 and Proposition 1.6.42. □

Proof of Proposition 1.6.44: Equation (1.6.23) is the definition(s) of the spaces $\lambda^{\#}_8$ and $\lambda^{\#}_9$ so there is nothing to prove. We have already proved the transversality of the sections $\Psi_{\mathcal{A}_0}$, $\Psi_{\mathcal{A}_1}$ and $\Psi_{\mathcal{D}_1}$ in Proposition 1.6.17. We will now show the transversality of the sections $\Psi_{\mathcal{A}_8}$, $\Psi_{\mathcal{E}_8}$ and $\Psi_{\mathcal{J}}$.

Let us start with $\Psi_{\mathcal{A}_8}$. Observe that by Lemma 1.4.18, statement 1 this section is well defined. Showing that this section is transverse to the zero set is equivalent to showing that the map

$$\bar{\Psi}_{\mathcal{A}_0} \oplus \bar{\Psi}_{\mathcal{A}_1} \oplus \bar{\Psi}_{\mathcal{D}_1} \oplus \bar{\Psi}_{\mathcal{A}_8} : \mathcal{F}^* \times \mathbb{C}^3 \rightarrow \mathbb{C}^{10},$$

$$(f, x, y, \eta) \mapsto (f(x, y), f_x, f_y, f_{xx}, f_{xy}, f_{yy}, f_{xxx}, f_{xxy}, f_{xyy}, f_{yyy})$$

is transverse to zero at $(f, 0, 0, 0)$. The Jacobian matrix of this map at $(f, 0, 0, 0)$ is a $10 \times (\delta + 4)$ matrix which has full rank if $d \geq 3$; take the first ten columns as the partial derivatives with respect to $f_{00}$, $f_{10}$, $f_{01}$, $f_{20}$, $f_{11}$, $f_{02}$, $f_{30}$, $f_{21}$, $f_{12}$ and $f_{03}$.

By Lemma 1.4.18, statement 1 the section $\Psi_{\mathcal{E}_8}$ is well defined. Showing that this section is transverse to the zero set is equivalent to showing that the map

$$\bar{\Psi}_{\mathcal{A}_0} \oplus \bar{\Psi}_{\mathcal{A}_1} \oplus \bar{\Psi}_{\mathcal{D}_1} \oplus \bar{\Psi}_{\mathcal{A}_8} \oplus \bar{\Psi}_{\mathcal{E}_8} : \mathcal{F}^* \times \mathbb{C}^3 \rightarrow \mathbb{C}^{11},$$

$$(f, x, y, \eta) \mapsto (f(x, y), f_x, f_y, f_{xx}, f_{xy}, f_{yy}, f_{xxx}, f_{xxy}, f_{xyy}, f_{yyy})$$

is transverse to zero at $(f, 0, 0, 0)$, where $\hat{x} = x + \eta y$. The Jacobian matrix of this map at $(f, 0, 0, 0)$ is an $11 \times (\delta + 4)$ matrix which has full rank if $d \geq 4$; take the eleven columns as the partial derivatives with respect to $f_{00}$, $f_{10}$, $f_{01}$, $f_{20}$, $f_{11}$, $f_{02}$, $f_{30}$, $f_{21}$, $f_{12}$, $f_{03}$ and $f_{40}$.

Finally, observe that by Lemma 1.4.18, statement 1 the section $\Psi_{\mathcal{J}}$ is well defined. Showing that this section is transverse to the zero set is equivalent to showing that the map

$$\bar{\Psi}_{\mathcal{A}_0} \oplus \bar{\Psi}_{\mathcal{A}_1} \oplus \bar{\Psi}_{\mathcal{D}_1} \oplus \bar{\Psi}_{\mathcal{A}_8} \oplus \bar{\Psi}_{\mathcal{E}_8} \oplus \bar{\Psi}_{\mathcal{J}} : \mathcal{F}^* \times \mathbb{C}^3 \rightarrow \mathbb{C}^{11},$$

$$(f, x, y, \eta) \mapsto (f(x, y), f_x, f_y, f_{xx}, f_{xy}, f_{yy}, f_{xxx}, f_{xxy}, f_{xyy}, f_{yyy}, -\frac{f_{xxxy}}{8} + \frac{3f_{xxxy}f_{xxyy} + 3f_{xxxy}f_{xxyy}}{16} - \frac{f_{xxxy}f_{xxyy}}{16})$$

is transverse to zero at $(f, 0, 0, 0)$, where $\hat{x} = x + \eta y$. The Jacobian matrix of this map at $(f, 0, 0, 0)$ is an $11 \times (\delta + 4)$ matrix which has full rank if $d \geq 4$ and $f_{40} \neq 0$. This follows by taking the first eleven columns to be the partial derivatives with respect to $f_{00}$, $f_{10}$, $f_{01}$, $f_{20}$, $f_{11}$, $f_{02}$, $f_{30}$, $f_{21}$, $f_{12}$, $f_{03}$ and $f_{13}$. Notice that the condition $\Psi_{\mathcal{E}_8}(\hat{f}, l) \neq 0$ (which is equivalent to $f_{40} \neq 0$) is necessary.
for the matrix to have full rank.

**Proof of Proposition 1.6.45:** The result for $\hat{X}_8^\#$ follows from Lemma 1.6.1 and Proposition 1.6.44 while that for $\hat{X}_8^\#9$ follows from Lemma 1.6.3 and Proposition 1.6.44.

### 1.7 Closure

We will now compute the closure of the various spaces. Along the way, we also compute the order to which certain sections vanish at a particular point. We supply the proofs of Lemma 1.7.1, (1) to (9), which were omitted in [1]. Moreover, the proofs of Lemma 1.7.1, (10) to (12) given in this manuscript are slightly more detailed than those given in [1].

**Lemma 1.7.1.** Let $X_k$ be a singularity of type $A_k$, $D_k$, $E_k$ or $X_8$. Then the closures are given by:

1. $\overline{A}_0 = A_0 \cup \overline{A}_1$ if $d \geq 2$.
2. $\overline{A}_1 = \overline{A}_1^\# = \hat{A}_1^\# \cup \overline{P \overline{A}}_2$ if $d \geq 3$.
3. $\overline{D}_4^\# = \overline{D}_4^\# \cup \overline{P \overline{D}}_4$ if $d \geq 3$.
4. $\overline{P \overline{D}}_4 = P \overline{D}_4 \cup \overline{P \overline{D}}_5 \cup \overline{P \overline{D}}_6$ if $d \geq 4$.
5. $\overline{P \overline{E}}_6 = P \overline{E}_6 \cup \overline{P \overline{E}}_7 \cup \overline{X}_8^\#$ if $d \geq 4$.
6. $\overline{P \overline{D}}_5 = P \overline{D}_5 \cup \overline{P \overline{D}}_6 \cup \overline{P \overline{E}}_6$ if $d \geq 4$.
7. $\overline{P \overline{D}}_6 = P \overline{D}_6 \cup \overline{P \overline{D}}_7 \cup \overline{P \overline{E}}_7$ if $d \geq 5$.
8. $\overline{P \overline{A}}_2 = P \overline{A}_2 \cup \overline{P \overline{A}}_3 \cup \overline{D}_4^\#$ if $d \geq 4$.
9. $\overline{P \overline{A}}_3 = P \overline{A}_3 \cup \overline{P \overline{A}}_4 \cup \overline{P \overline{D}}_4$ if $d \geq 5$.
10. $\overline{P \overline{A}}_4 = P \overline{A}_4 \cup \overline{P \overline{A}}_5 \cup \overline{P \overline{D}}_5$ if $d \geq 6$.
11. $\overline{P \overline{A}}_5 = P \overline{A}_5 \cup \overline{P \overline{A}}_6 \cup \overline{P \overline{D}}_6 \cup \overline{P \overline{E}}_6$ if $d \geq 7$.
12. $\overline{P \overline{A}}_6 = P \overline{A}_6 \cup \overline{P \overline{A}}_7 \cup \overline{P \overline{D}}_7 \cup \overline{P \overline{E}}_7 \cup \overline{X}_8^\#$ if $d \geq 8$.

**Proof of Lemma 1.7.1 (1):** Follows from Corollary 1.6.5, Proposition 1.6.4, Corollary 1.6.7.

**Proof of Lemma 1.7.1 (2):** Follows from Corollary 1.6.12, 1.6.14, Proposition 1.6.13, Corollary 1.6.16.

**Proof of Lemma 1.7.1 (3):** Follows from Corollary 1.6.18, Proposition 1.6.17, Corollary 1.6.20, 1.6.16, 1.6.14 and 1.6.10.

**Proof of Lemma 1.7.1 (4):** Follows from Corollary 1.6.20, Proposition 1.6.19, Corollary 1.6.27 and 1.6.29.
Proof of Lemma 1.7.1 (5): Corollary 1.6.45 implies that

\[ \tilde{X}_8^\# = \{ (\tilde{f}, l_{\tilde{p}}) \in D \times \mathbb{PTP}^2 : \Psi_{A_0}(\tilde{f}, l_{\tilde{p}}) = 0, \Psi_{A_1}(\tilde{f}, l_{\tilde{p}}) = 0, \Psi_{D_4}(\tilde{f}, l_{\tilde{p}}) = 0, \Psi_{X_8}(\tilde{f}, l_{\tilde{p}}) = 0 \}. \]

Standard linear algebra implies that

\[ \tilde{X}_8^\# = \{ (\tilde{f}, l_{\tilde{p}}) \in D \times \mathbb{PTP}^2 : \Psi_{A_0}(\tilde{f}, l_{\tilde{p}}) = 0, \Psi_{A_1}(\tilde{f}, l_{\tilde{p}}) = 0, \Psi_{P_{\mathbb{A}_2}}(\tilde{f}, l_{\tilde{p}}) = 0, \Psi_{P_{\mathbb{A}_3}}(\tilde{f}, l_{\tilde{p}}) = 0, \Psi_{P_{\mathbb{X}_8}}(\tilde{f}, l_{\tilde{p}}) = 0 \}. \]

By Corollary 1.6.32, 1.6.27, 1.6.20, 1.6.16, 1.6.14, 1.6.10, we conclude that

\[ \tilde{X}_8^\# = \{ (\tilde{f}, l_{\tilde{p}}) \in \mathbb{PTP}^2 : \Psi_{X_8}(\tilde{f}, l_{\tilde{p}}) = 0 \}. \] (1.7.1)

Lemma 1.7.1, statement 5 now follows from (1.7.1), Corollary 1.6.40, Proposition 1.6.31 and Corollary 1.6.32.

Proof of Lemma 1.7.1 (6): Follows from Corollary 1.6.27, Proposition 1.6.26, Corollary 1.6.35 and 1.6.32.

Proof of Lemma 1.7.1 (7): Follows from Proposition 1.6.37. Since the section

\[ \Psi_{E_6} : \mathbb{PD}_6 \rightarrow \mathbb{E}_6 \]

is transverse to the zero set (as proved in Proposition 1.6.34), the hypothesis of the second part of Lemma 1.6.2 is satisfied. By Proposition 1.6.34 and Corollary 1.6.35 we conclude that

\[ \mathbb{PD}_6 = \{ (\tilde{f}, l_{\tilde{p}}) \in \mathbb{PD}_6 : \Psi_{PD_7}(\tilde{f}, l_{\tilde{p}}) \neq 0, \Psi_{P_{E_6}}(\tilde{f}, l_{\tilde{p}}) \neq 0 \}. \]

Corollary 1.6.43 now proves our claim.

Proof of Lemma 1.7.1 (8): Follows from Corollary 1.6.16, Proposition 1.6.15, Corollary 1.6.22, 1.6.18, 1.6.14 and 1.6.10.

Proof of Lemma 1.7.1 (9): Follows from the second part of Lemma 1.6.2 and Proposition 1.6.24. Since the section \( \Psi_{PD_4} : \mathbb{PA}_4 \rightarrow \mathbb{PD}_4 \) is transverse to the zero set (cf. Proposition 1.6.21), the hypothesis of the second part of Lemma 1.6.2 is satisfied. Corollary 1.6.20 and 1.6.22 now imply that

\[ \mathbb{PA}_3 = \{ (\tilde{f}, l_{\tilde{p}}) \in \mathbb{PA}_3 : \Psi_{PA_4}(\tilde{f}, l_{\tilde{p}}) \neq 0, \Psi_{PD_4}(\tilde{f}, l_{\tilde{p}}) \neq 0 \}. \] (1.7.2)

By Proposition 1.6.21 and Corollary 1.6.22 we get that

\[ \mathbb{PA}_3 = \{ (\tilde{f}, l_{\tilde{p}}) \in \mathbb{PA}_3 : \Psi_{PA_4}(\tilde{f}, l_{\tilde{p}}) \neq 0, \Psi_{PD_4}(\tilde{f}, l_{\tilde{p}}) \neq 0 \}. \]

The second part of Lemma 1.6.2 now proves our claim.

Proof of Lemma 1.7.1 (10): By Lemma 1.6.2 applied to

\[ M = \mathbb{PA}_3, \quad \zeta_0 = \Psi_{PA_4}, \quad \zeta_1 = \Psi_{PA_5}, \quad \zeta_2 = \Psi_{PA_6}, \quad \varphi = \Psi_{PD_4}, \]

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and Proposition 1.6.24, it suffices to show that
\[
\{(\tilde{f}, \tilde{l}_p) \in \overline{\mathcal{A}_4} : \Psi_{PD_4}(\tilde{f}, \tilde{l}_p) = 0\} = \overline{\mathcal{D}}_5.
\]
(1.7.3)

Let us show that the left hand side of (1.7.3) is a subset of the right hand side. We claim that
\[
\overline{\mathcal{A}_4} \cap \mathcal{D}_4 = \emptyset.
\]
(1.7.4)

To see this, first we observe that if \((\tilde{f}, \tilde{l}_p) \in \mathcal{D}_4\) then \(\Psi_{PD_4}(\tilde{f}, \tilde{l}_p) = 0\) and \(\Psi_{PD_5}(\tilde{f}, \tilde{l}_p) \neq 0\). Therefore,
\[
\Psi_{PA_4}(\tilde{f}, \tilde{l}_p) = \Psi_{PD_4}(\tilde{f}, \tilde{l}_p)\Psi_{PD_5}(\tilde{f}, \tilde{l}_p) - 3\Psi_{PD_5}(\tilde{f}, \tilde{l}_p)^2 = -3\Psi_{PD_5}(\tilde{f}, \tilde{l}_p)^2 \neq 0.
\]

This implies that if \((\tilde{f}(t), l_{\tilde{p}}(t))\) lies in a small neighborhood of \((\tilde{f}, \tilde{l}_p)\) then \(\Psi_{PA_4}(\tilde{f}(t), l_{\tilde{p}}(t)) \neq 0\), proving (1.7.4). By Lemma 1.7.1, statement 9 we have \(\overline{\mathcal{A}_4} \subset \overline{\mathcal{A}_3}\). Therefore,
\[
\{(\tilde{f}, \tilde{l}_p) \in \overline{\mathcal{A}_4} : \Psi_{PD_4}(\tilde{f}, \tilde{l}_p) = 0\} \subset \{(\tilde{f}, \tilde{l}_p) \in \overline{\mathcal{A}_3} : \Psi_{PD_4}(\tilde{f}, \tilde{l}_p) = 0\}.
\]

The right hand side above equals
\[
\mathcal{P}_4 \cup \overline{\mathcal{D}_5}
\]
by (1.7.2) and Lemma 1.7.1, statement 4. Hence,
\[
\{(\tilde{f}, \tilde{l}_p) \in \overline{\mathcal{A}_4} : \Psi_{PD_4}(\tilde{f}, \tilde{l}_p) = 0\} \subset \overline{\mathcal{D}_5}
\]
by (1.7.4). This proves that the left hand side of (1.7.3) is a subset of the right hand side. For the converse note since \(\overline{\mathcal{A}_4}\) is a closed set, it suffices to show that
\[
\{(\tilde{f}, \tilde{l}_p) \in \overline{\mathcal{A}_4} : \Psi_{PD_4}(\tilde{f}, \tilde{l}_p) = 0\} \supset \mathcal{D}_5.
\]
(1.7.5)

We will simultaneously prove statement (1.7.5) and also prove the following statement
\[
\overline{\mathcal{A}_5} \cap \mathcal{D}_5 = \emptyset.
\]
(1.7.6)

Since \(\mathcal{D}_5\) and \(\mathcal{A}_5\) are both subsets of \(\overline{\mathcal{A}_3}\), we can consider closures inside \(\overline{\mathcal{A}_3}\).

**Claim 1.7.2.** Let \((\tilde{f}, \tilde{l}_p) \in \mathcal{D}_5\). Then there exists a solution \((\tilde{f}(t), l_{\tilde{p}}(t)) \in \overline{\mathcal{A}_3}\) near \((\tilde{f}, \tilde{l}_p)\) to the set of equations
\[
\Psi_{PD_4}(\tilde{f}(t), l_{\tilde{p}}(t)) \neq 0, \quad \Psi_{PA_4}(\tilde{f}(t), l_{\tilde{p}}(t)) = 0.
\]
(1.7.7)

Moreover, whenever such a solution \((\tilde{f}(t), l_{\tilde{p}}(t))\) is sufficiently close to \((\tilde{f}, \tilde{l}_p)\) it lies in \(\mathcal{A}_4\), i.e., \(\Psi_{PA_5}(\tilde{f}(t), l_{\tilde{p}}(t)) \neq 0\). In particular \((\tilde{f}(t), l_{\tilde{p}}(t))\) does not lie in \(\mathcal{A}_5\).

It is easy to see that claim 1.7.2 proves statements (1.7.5) and (1.7.6) simultaneously.

**Proof:** Let \(v \in \tilde{\gamma}, \tilde{w} \in \pi^*TP^2/\tilde{\gamma}\) and \(f_{ij}\) be as defined in (1.3.5), subsection 1.3.2. Equation (1.7.7) is a functional equation since the quantities \(\Psi_{PD_4}(\tilde{f}(t), l_{\tilde{p}}(t))\) and \(\Psi_{PA_4}(\tilde{f}(t), l_{\tilde{p}}(t))\) are functionals, i.e., they act on vectors \(v\) and \(\tilde{w}\) and produce a number. We will first solve the corresponding equation
\[
\{(\Psi_{PD_4}(\tilde{f}(t), l_{\tilde{p}}(t)))(p \otimes f_{ij}^d \otimes \tilde{w}^\otimes 2) = f_{02}(t) \neq 0
\}
\]
\[
\{(\Psi_{PA_4}(\tilde{f}(t), l_{\tilde{p}}(t)))(p \otimes f_{ij}^d \otimes v^\otimes 2 \otimes \tilde{w}) = f_{02}(t)A_4^{f(t)} = 0.
\]
(1.7.8)
In (1.7.8) equality holds as numbers. It is easy to see that the only solutions to (1.7.8) are of the form

\[ f_{21}(t) = u, \quad f_{02}(t) = \frac{3u^2}{f_{40}(t)}. \]  

(1.7.9)

Equation (1.7.9) implies that the only solutions to the functional equation (1.7.7) is of the form

\[ \Psi_{PD_5}^L(\tilde{f}(t), l_{\tilde{p}}(t)) = t, \quad \Psi_{PD_4}(\tilde{f}(t), l_{\tilde{p}}(t)) = \frac{3t^2}{\Psi_{PD_6}(\tilde{f}(t), l_{\tilde{p}}(t))}. \]  

(1.7.10)

where equality holds as functionals.

**Remark 1.7.3.** To avoid confusion, let us explain our notation carefully. The notation \((\tilde{f}(t), l_{\tilde{p}}(t))\) is simply used to indicate that \((\tilde{f}(t), l_{\tilde{p}}(t))\) is some point sufficiently close to \((\tilde{f}, l_{\tilde{p}})\). We denote the functional \(\Psi_{PD_5}^L(\tilde{f}(t), l_{\tilde{p}}(t))\) by the letter \(t\). Hence, \(t\) is close to the zero functional, since \(\Psi_{PD_5}^L(\tilde{f}, l_{\tilde{p}}) = 0\). Next, we denote the number

\[ \{\Psi_{PD_5}^L(\tilde{f}(t), l_{\tilde{p}}(t))\}(v^\otimes 2 \otimes \tilde{w}) \]

by the symbol \(f_{21}(t)\). This number is close to the number zero. We also need to denote this number with some symbol. We decided to use the symbol \(u\). Hence, we have this seemingly awkward equation \(f_{21}(t) = u\).

To summarize, \(t\) is functional, while \(u\) is a number. Now comes a crucial observation: the sections

\[ \Psi_{PD_4} : \overline{PA}_3 \rightarrow \mathbb{L}_{PD_4} \quad \text{and} \quad \Psi_{PD_5}^L : \Psi_{PD_4}^{-1}(0) \rightarrow \mathbb{L}_{PD_5} \]

are transverse to the zero set (cf. Proposition 1.6.22). Since \(\Psi_{PD_4}^{-1}(0)\) is a smooth manifold, we can extend the section \(\Psi_{PD_5}^L\) outside a small neighborhood of \(\Psi_{PD_4}^{-1}(0)\) using the exponential map (recall that \(\Psi_{PD_5}^L\) is well defined only on \(\Psi_{PD_4}^{-1}(0)\)). Therefore, there exists a solution \((\tilde{f}(t), l_{\tilde{p}}(t))\) close to \((\tilde{f}, l_{\tilde{p}})\) to (1.7.10). This proves our first assertion. Now we need to show that any such solution satisfies the condition \(\Psi_{PA_5}(\tilde{f}(t), l_{\tilde{p}}(t)) \neq 0\) if \(t\) is sufficiently small. Observe that

\[ f_{02}(t)^2 A_5^{(t)} = 15f_{12}(t)u^2 + O(u^3) \quad \text{using (1.7.9).} \]

\[ \Rightarrow \Psi_{PA_5}(\tilde{f}(t), l_{\tilde{p}}(t)) = 15\Psi_{PA_6}(\tilde{f}(t), l_{\tilde{p}}(t))t^2 + O(t^3) \]  

(1.7.11)

Since \((\tilde{f}, l_{\tilde{p}}) \in PD_5\), we get that \(\Psi_{PA_6}(\tilde{f}, l_{\tilde{p}}) \neq 0\) (cf. Proposition 1.6.26). Hence, by (1.7.11), if \(t\) is sufficiently small then \(\Psi_{PA_5}(\tilde{f}(t), l_{\tilde{p}}(t)) \neq 0\). This proves claim 1.7.2. \(\square\)

Before proving the next Lemma, we prove a corollary which follows immediately from the previous discussion.

**Corollary 1.7.4.** Let \(\mathbb{W} \rightarrow D \times \mathbb{P}\mathbb{P}^2\) be a vector bundle such that the rank of \(\mathbb{W}\) is same as the dimension of \(PD_5\) and \(Q : D \times \mathbb{P}\mathbb{P}^2 \rightarrow \mathbb{W}\) a generic smooth section. Suppose \((\tilde{f}, l_{\tilde{p}}) \in PD_5 \cap Q^{-1}(0)\). Then the section

\[ \Psi_{PA_5} \oplus Q : \overline{PA}_4 \rightarrow \mathbb{L}_{PA_5} \oplus \mathbb{W} \]

vanishes around \((\tilde{f}, l_{\tilde{p}})\) with a multiplicity of 2.
Proof: Since the section $Q$ is generic, $Q^{-1}(0)$ intersects $\mathcal{P}D_5$ transversely. Since the rank of $\mathcal{W}$ is equal to the dimension of $\mathcal{P}D_5$ there exists a unique solution $(\tilde{f}(t), l_p(t)) \in \overline{\mathcal{A}}_3$ near $(f, l_p)$ to the set of equations

$$\Psi_{\mathcal{P}D_5}^L(\tilde{f}(t), l_p(t)) = t, \quad \Psi_{\mathcal{P}D_4}(\tilde{f}(t), l_p(t)) = \frac{3t^2}{\Psi_{\mathcal{P}D_6}(\tilde{f}(t), l_p(t))}, \quad Q(\tilde{f}(t), l_p(t)) = 0.$$  

The claim follows from (1.7.10) combined with the added condition $Q(\tilde{f}(t), l_p(t)) = 0$. Equation (1.7.11) now proves our claim, since $\Psi_{\mathcal{P}\mathcal{E}_6}(\tilde{f}, l_p) \neq 0$. \hfill \qedsymbol

Remark 1.7.5. This idea is due to Aleksey Zinger - the crucial observation that we can use the transversality of the bundle sections to describe the neighborhood of a point.

Proof of Lemma 1.7.1 (11): By Lemma 1.6.2 applied to

$$M = \overline{\mathcal{A}}_3, \quad \zeta_0 = \Psi_{\mathcal{P}\mathcal{A}_4}, \quad \zeta_1 = \Psi_{\mathcal{P}\mathcal{A}_5}, \quad \zeta_2 = \Psi_{\mathcal{P}\mathcal{A}_6}, \quad \zeta_3 = \Psi_{\mathcal{P}\mathcal{A}_7}, \quad \varphi = \Psi_{\mathcal{P}D_4},$$

and Proposition 1.6.24, it suffices to show that

$$\{(\tilde{f}, l_p) \in \overline{\mathcal{A}}_5 : \Psi_{\mathcal{P}D_4}(\tilde{f}, l_p) = 0\} = \overline{\mathcal{D}}_6 \cup \overline{\mathcal{E}}_6. \quad (1.7.12)$$

We will do this in two steps. We will show that

$$\{(\tilde{f}, l_p) \in \overline{\mathcal{A}}_5 : \Psi_{\mathcal{P}D_4}(\tilde{f}, l_p) = 0, \Psi_{\mathcal{P}\mathcal{E}_6}(\tilde{f}, l_p) \neq 0\} \cup \{(\tilde{f}, l_p) \in \overline{\mathcal{D}}_6 : \Psi_{\mathcal{P}\mathcal{E}_6}(\tilde{f}, l_p) = 0\} \quad (1.7.13)$$

and

$$\{(\tilde{f}, l_p) \in \overline{\mathcal{A}}_5 : \Psi_{\mathcal{P}D_4}(\tilde{f}, l_p) = 0, \Psi_{\mathcal{P}\mathcal{E}_6}(\tilde{f}, l_p) = 0\} = \overline{\mathcal{E}}_6. \quad (1.7.14)$$

It follows that (1.7.13) and (1.7.14) imply (1.7.12). To see this, note that

$$\{(\tilde{f}, l_p) \in \overline{\mathcal{D}}_6 : \Psi_{\mathcal{P}\mathcal{E}_6}(\tilde{f}, l_p) = 0\} \subset \overline{\mathcal{E}}_6 \quad \text{(1.7.15)}$$

It is now easy to see that (1.7.15), (1.7.13) and (1.7.14) imply (1.7.12).

We will now start with the proof of (1.7.13). We will show that the left hand side of (1.7.13) is a subset of the right hand side. This follows from (1.7.6). By Lemma 1.7.1, statement 10 we know that $\overline{\mathcal{A}}_5 \subset \overline{\mathcal{A}}_4$. Therefore, in conjunction with (1.7.3), we gather that

$$\{(\tilde{f}, l_p) \in \overline{\mathcal{A}}_5 : \Psi_{\mathcal{P}D_4}(\tilde{f}, l_p) = 0\} \subset \{(\tilde{f}, l_p) \in \overline{\mathcal{A}}_4 : \Psi_{\mathcal{P}D_4}(\tilde{f}, l_p) = 0\} = \overline{\mathcal{D}}_5. \quad (1.7.16)$$

The right hand side above equals $\mathcal{P}D_5 \cup \overline{\mathcal{D}}_6 \cup \overline{\mathcal{E}}_6$ by Lemma 1.7.1, statement 6. Hence, by (1.7.6)

$$\{(\tilde{f}, l_p) \in \overline{\mathcal{A}}_5 : \Psi_{\mathcal{P}D_4}(\tilde{f}, l_p) = 0, \Psi_{\mathcal{P}\mathcal{E}_6}(\tilde{f}, l_p) \neq 0\} \subset \{(\tilde{f}, l_p) \in \overline{\mathcal{D}}_6 : \Psi_{\mathcal{P}\mathcal{E}_6}(\tilde{f}, l_p) \neq 0\} \quad \text{(1.7.17)}$$

Now we will show the converse. Since $\overline{\mathcal{A}}_5$ is a closed space, it suffices to show that

$$\{(\tilde{f}, l_p) \in \overline{\mathcal{A}}_5 : \Psi_{\mathcal{P}D_4}(\tilde{f}, l_p) = 0, \Psi_{\mathcal{P}\mathcal{E}_6}(\tilde{f}, l_p) \neq 0\} \supset \{(\tilde{f}, l_p) \in \mathcal{P}D_6 : \Psi_{\mathcal{P}\mathcal{E}_6}(\tilde{f}, l_p) \neq 0\}.$$  

As before, we will simultaneously prove (1.7.16) and also prove that

$$\overline{\mathcal{A}}_5 \cap \mathcal{P}D_6 = \emptyset. \quad (1.7.17)$$
Claim 1.7.6. Let \((\tilde{f}, l_{\tilde{p}}) \in \mathcal{PD}_6\). Then there exists a solution \((\tilde{f}(t), l_{\tilde{p}}(t)) \in \mathcal{PA}_3\) near \((\tilde{f}, l_{\tilde{p}})\) to the set of equations

\[
\Psi_{\mathcal{PD}_4}(\tilde{f}(t), l_{\tilde{p}}(t)) \neq 0, \quad \Psi_{\mathcal{PA}_4}(\tilde{f}(t), l_{\tilde{p}}(t)) = 0, \quad \Psi_{\mathcal{PA}_5}(\tilde{f}(t), l_{\tilde{p}}(t)) = 0. \tag{1.7.18}
\]

Moreover, whenever such a solution \((\tilde{f}(t), l_{\tilde{p}}(t))\) is sufficiently close to \((\tilde{f}, l_{\tilde{p}})\) it lies in \(\mathcal{PA}_5\), i.e., \(\Psi_{\mathcal{PA}_6}(\tilde{f}(t), l_{\tilde{p}}(t)) \neq 0\). In particular, \((\tilde{f}(t), l_{\tilde{p}}(t))\) does not lie in \(\mathcal{PA}_6\).

It is clear that claim 1.7.6 proves (1.7.16) and (1.7.17) simultaneously.

Proof: As before, we will first solve the equation

\[
f_{02}(t) \neq 0, \quad f_{02}(t)A^f_4 = 0, \quad f_{02}(t)^2A^f_5 = 0. \tag{1.7.19}
\]

The only solutions to (1.7.19) are of the form

\[
f_{02}(t) = u, \\
f_{21}(t) = \left(\frac{5f_{31}(t) \pm \sqrt{-15f_{12}(t)D^f_7(t)}}{15f_{12}(t)}\right) u, \\
f_{40}(t) = 3\left(\frac{5f_{31}(t) \pm \sqrt{-15f_{12}(t)D^f_7(t)}}{15f_{12}(t)}\right)^2 u. \tag{1.7.20}
\]

The second equation comes from solving a quadratic arising from \(f_{02}(t)^2A^f_5 = 0\) while the third is from solving \(f_{02}(t)A^f_4 = 0\) and using \(f_{21}\) from the second equation. Since \((\tilde{f}, l_{\tilde{p}}) \in \mathcal{PD}_6\) we know that \(f_{12} \neq 0\) and \(D^f_7 \neq 0\). Hence, there are always two solutions for \((f_{21}(t), f_{40}(t))\). Equation (1.7.20) implies that the only solutions to the functional equation (1.7.18) is of the form

\[
\Psi_{\mathcal{PD}_4}(\tilde{f}(t), l_{\tilde{p}}(t)) = t \\
\Psi^L_{\mathcal{PD}_5}(\tilde{f}(t), l_{\tilde{p}}(t)) = \left(\frac{5\Psi_{\mathcal{PE}_5}(\tilde{f}(t), l_{\tilde{p}}(t)) \pm \sqrt{-15\Psi_{\mathcal{PE}_6}(\tilde{f}(t), l_{\tilde{p}}(t))\Psi_{\mathcal{PD}_7}(\tilde{f}(t), l_{\tilde{p}}(t))}}{15\Psi_{\mathcal{PE}_6}(\tilde{f}(t), l_{\tilde{p}}(t))}\right) t \tag{1.7.21}
\]

where equality holds as functionals. Since the sections

\[
\Psi_{\mathcal{PD}_4} : \mathcal{PA}_3 \longrightarrow \mathbb{L}_{\mathcal{PD}_4}, \quad \Psi^L_{\mathcal{PD}_5} : \Psi^{-1}_{\mathcal{PD}_4}(0) \longrightarrow \mathbb{L}_{\mathcal{PD}_5} \quad \text{and} \quad \Psi_{\mathcal{PD}_6} : \Psi^{-1}_{\mathcal{PD}_5}(0) \longrightarrow \mathbb{L}_{\mathcal{PD}_6}
\]

are transverse to the zero set (as proved in Proposition 1.6.19 and 1.6.26), there exists a solution \((\tilde{f}(t), l_{\tilde{p}}(t))\) close to \((\tilde{f}, l_{\tilde{p}})\) to (1.7.21). This proves our first assertion. Next we need to show that any such solution satisfies the condition \(\Psi_{\mathcal{PA}_6}(\tilde{f}(t), l_{\tilde{p}}(t)) \neq 0\) if \(t\) is sufficiently small. To prove that we observe

\[
f_{02}(t)^3A^f_3 = \frac{D^f_7(t)}{f_{12}(t)} u^2 + O(u^3) \quad \text{using (1.7.20), for each choice of } \sqrt{f_{12}D^f_7(t)}. \tag{1.7.22}
\]

\[
\Rightarrow \quad \Psi_{\mathcal{PA}_6}(\tilde{f}(t), l_{\tilde{p}}(t)) = \frac{\Psi_{\mathcal{PD}_7}(\tilde{f}(t), l_{\tilde{p}}(t))}{\Psi_{\mathcal{PE}_6}(\tilde{f}(t), l_{\tilde{p}}(t))^2} t^2 + O(t^3)
\]
Since \((\bar{f},l_{\bar{p}}) \in \mathcal{PD}_6\), we get that \(\Psi_{\mathcal{PE}}(\bar{f},l_{\bar{p}}), \Psi_{\mathcal{PD}}(\bar{f},l_{\bar{p}}) \neq 0\) (see Proposition 1.6.34). Hence, (1.7.22) implies that if \(t\) is sufficiently small \(\Psi_{\mathcal{PA}}(\bar{f}(t),l_{\bar{p}}(t)) \neq 0\), which proves claim 1.7.6.

Before proving (1.7.14), we prove a corollary which follows immediately.

**Corollary 1.7.7.** Let \(\mathbb{W} \to \mathcal{D} \times \mathbb{P}^2\) be a vector bundle such that the rank of \(\mathbb{W}\) is same as the dimension of \(\mathcal{PD}_6\). Let \(\mathcal{Q} : \mathcal{D} \times \mathbb{P}^2 \to \mathbb{W}\) be a generic smooth section. Suppose \((\bar{f},l_{\bar{p}}) \in \mathcal{PD}_6 \cap \mathcal{Q}^{-1}(0)\). Then the section

\[
\Psi_{\mathcal{PA}} \oplus \mathcal{Q} : \mathcal{PA}_5 \to \mathbb{L}_{\mathcal{PA}_6} \oplus \mathbb{W}
\]

vanishes around \((\bar{f},l_{\bar{p}})\) with a multiplicity of 4.

**Proof:** As before, in the proof of claim 1.7.4, this follows from (1.7.22) and the fact that \(\mathcal{Q}^{-1}(0)\) intersects \(\mathcal{PD}_6\) transversely. Each branch of \(\sqrt{\mathbb{f}_d\mathbb{D}_l^I}\) contributes with a multiplicity of 2. Hence, the total multiplicity is 4. \(\square\)

Next we will prove (1.7.14). We will show that the left hand side is a subset of the right hand side. Note that \(\mathcal{PA}_5 \subset \mathcal{PA}_4\) is implied by Lemma 1.7.1, statement 10, whence

\[
\{(\bar{f},l_{\bar{p}}) \in \mathcal{PA}_5 : \Psi_{\mathcal{PD}_4}(\bar{f},l_{\bar{p}}) = 0, \Psi_{\mathcal{PE}_6}(\bar{f},l_{\bar{p}}) = 0\} \subset \{(\bar{f},l_{\bar{p}}) \in \mathcal{PA}_4 : \Psi_{\mathcal{PD}_4}(\bar{f},l_{\bar{p}}) = 0, \Psi_{\mathcal{PE}_6}(\bar{f},l_{\bar{p}}) = 0\}.
\]

The right hand side above can be simplified (1.7.3) and Corollary 1.6.32 as follows:

\[
\{(\bar{f},l_{\bar{p}}) \in \mathcal{PA}_4 : \Psi_{\mathcal{PD}_4}(\bar{f},l_{\bar{p}}) = 0, \Psi_{\mathcal{PE}_6}(\bar{f},l_{\bar{p}}) = 0\} = \{(\bar{f},l_{\bar{p}}) \in \mathcal{PA}_5 : \Psi_{\mathcal{PD}_4}(\bar{f},l_{\bar{p}}) = 0, \Psi_{\mathcal{PE}_6}(\bar{f},l_{\bar{p}}) = 0\} = \mathcal{PE}_6.
\]

This implies

\[
\{(\bar{f},l_{\bar{p}}) \in \mathcal{PA}_5 : \Psi_{\mathcal{PD}_4}(\bar{f},l_{\bar{p}}) = 0, \Psi_{\mathcal{PE}_6}(\bar{f},l_{\bar{p}}) = 0\} \subset \mathcal{PE}_6.
\]

Now we prove the converse. Since \(\mathcal{PA}_5\) is a closed set, it suffices to show that

\[
\{(\tilde{f},l_{\tilde{p}}) \in \mathcal{PA}_5 : \Psi_{\mathcal{PD}_4}(\tilde{f},l_{\tilde{p}}) = 0, \Psi_{\mathcal{PE}_6}(\tilde{f},l_{\tilde{p}}) = 0\} \supset \mathcal{PE}_6.
\]

As before, we will simultaneously prove (1.7.23) and also prove the following:

\[
\mathcal{PA}_6 \cap \mathcal{PE}_6 = \emptyset.
\]

**Claim 1.7.8.** Let \((\tilde{f},l_{\tilde{p}}) \in \mathcal{PE}_6\). Then there exists a solution \((\tilde{f}(t),l_{\tilde{p}}(t)) \in \mathcal{PA}_3\) near \((\tilde{f},l_{\tilde{p}})\) to the set of equations

\[
\Psi_{\mathcal{PD}_4}(\tilde{f}(t),l_{\tilde{p}}(t)) \neq 0, \quad \Psi_{\mathcal{PA}_4}(\tilde{f}(t),l_{\tilde{p}}(t)) = 0, \quad \Psi_{\mathcal{PA}_5}(\tilde{f}(t),l_{\tilde{p}}(t)) = 0.
\]

Moreover, whenever such a solution \((\tilde{f}(t),l_{\tilde{p}}(t))\) is sufficiently close to \((\tilde{f},l_{\tilde{p}})\), it lies in \(\mathcal{PA}_5\), i.e., \(\Psi_{\mathcal{PA}_6}(\tilde{f}(t),l_{\tilde{p}}(t)) \neq 0\). In particular \((\tilde{f}(t),l_{\tilde{p}}(t))\) does not lie in \(\mathcal{PA}_6\).

Note that claim 1.7.8 proves (1.7.23) and (1.7.24) simultaneously.

**Proof:** As before, we will first solve the equations

\[
f_{02}(t) \neq 0, \quad f_{02}(t)A_{4}^{f(t)} = 0, \quad f_{02}(t)A_{5}^{f(t)} = 0.
\]
It is easy to see that the only solutions to \((1.7.26)\) are of the form
\[
\begin{align*}
    f_{21}(t) &= u, & f_{02} &= \frac{3u^2}{f_{40}(t)}, & \text{and} & f_{12} &= \frac{2f_{31}(t)}{f_{40}(t)} u - \frac{3f_{50}(t)}{5f_{40}(t)^2} u^2. \\
\end{align*}
\]
(1.7.27)

Note that since \((\hat{f}, \hat{l}_p) \in PE_6\) we get that \(f_{40} \neq 0\). Equation \((1.7.27)\) implies that the only solutions to the functional equation \((1.7.25)\) is of the form
\[
\begin{align*}
    \Psi_{PD_6}(\hat{f}(t), \hat{l}_p(t)) &= t \\
    \Psi_{PD_4}(\hat{f}(t), \hat{l}_p(t)) &= \frac{3t^2}{\Psi_{PD_6}(\hat{f}(t), \hat{l}_p(t))} \\
    \Psi_{PE_6}(\hat{f}(t), \hat{l}_p(t)) &= \frac{2\Psi_{PE_4}(\hat{f}(t), \hat{l}_p(t))}{\Psi_{PD_6}(\hat{f}(t), \hat{l}_p(t))} t + O(t^2)
\end{align*}
\]
(1.7.28)

where equality holds as functionals. Since the sections
\[
\Psi_{PD_4} : \overline{PA}_3 \longrightarrow L_{PD_4}, \quad \Psi_{PD_5}^1 : \Psi_{PD_5}^{-1}(0) \longrightarrow L_{PD_5} \quad \text{and} \quad \Psi_{PE_6} : \Psi_{PD_5}^{-1}(0) \longrightarrow L_{PE_6}
\]
are transverse to the zero set (as proved in Proposition 1.6.19 and 1.6.26), there exists a solution \((\hat{f}(t), \hat{l}_p(t))\) close to \((\hat{f}, \hat{l}_p)\) to \((1.7.28)\). This proves our first assertion. Next we need to show that any such solution satisfies the condition \(\Psi_{PA_6}(\hat{f}(t), \hat{l}_p(t)) \neq 0\) if \(t\) is sufficiently small. To prove that we observe
\[
\begin{align*}
    f_{02}(t)^3 A_6^{f(t)} &= -15 f_{03}(t) u^3 + O(u^4) \quad \text{using} \ (1.7.27). \\
    \implies \Psi_{PA_6}(\hat{f}(t), \hat{l}_p(t)) &= -15 \Psi_{PA_6}(\hat{f}(t), \hat{l}_p(t)) t^3 + O(t^3)
\end{align*}
\]
(1.7.29)

Since \((\hat{f}, \hat{l}_p) \in PE_6\), we get that \(\Psi_{PA_6}(\hat{f}(t), \hat{l}_p(t)) \neq 0\) (see Proposition 1.6.31). Hence, \((1.7.29)\) implies that if \(t\) is sufficiently small \(\Psi_{PA_6}(\hat{f}(t), \hat{l}_p(t)) \neq 0\) which proves claim 1.7.8

\[\square\]

**Corollary 1.7.9.** Let \(W \longrightarrow D \times \mathbb{P}\mathbb{P}^2\) be a vector bundle such that the rank of \(W\) is same as the dimension of \(PE_6\). Let \(Q : D \times \mathbb{P}\mathbb{P}^2 \longrightarrow W\) be a generic smooth section. Suppose \((\hat{f}, \hat{l}_p) \in PE_6 \cap Q^{-1}(0)\). Then the section
\[
\Psi_{PA_6} \oplus Q : \overline{PA}_5 \longrightarrow L_{PA_6} \oplus W
\]
vanishes around \((\hat{f}, \hat{l}_p)\) with a multiplicity of 3.

**Proof:** Follows from the fact that \(Q^{-1}(0)\) intersects \(PE_6\) transversely and \((1.7.29)\).

\[\square\]

**Proof of Lemma 1.7.1 (12):** First of all we note that it suffices to show that
\[
\overline{PA}_6 = PA_6 \cup \overline{PA}_7 \cup \overline{PD}_7 \cup \overline{PE}_7 \cup \overline{X}_8^{\#5}.
\]
This is because, by Corollary 1.6.45
\[
\overline{X}_8^{\#5} = \overline{X}_8^{\#}. \quad \text{By Lemma 1.6.2 applied to}
\]
\[
M = \overline{PA}_3, \quad \zeta_0 = \Psi_{PA_4}, \quad \zeta_1 = \Psi_{PA_5}, \quad \zeta_2 = \Psi_{PA_6}, \quad \zeta_3 = \Psi_{PA_6}, \quad \zeta_4 = \Psi_{PA_6}, \quad \varphi = \Psi_{PD_4},
\]
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and Proposition 1.6.24, it suffices to show that

\[
\{(\hat{f}, l_{\hat{p}}) \in \overline{\mathcal{PA}_6} : \Psi_{PD_4}(\hat{f}, l_{\hat{p}}) = 0\} = \overline{PD_7} \cup \overline{PE_7} \cup \overline{\mathcal{X}^#_8}. \tag{1.7.30}
\]

We will do this in three steps. We will show that

\[
\begin{align*}
\{(\hat{f}, l_{\hat{p}}) \in \overline{\mathcal{PA}_6} : & \Psi_{PD_4}(\hat{f}, l_{\hat{p}}) = 0, \ \Psi_{PE_6}(\hat{f}, l_{\hat{p}}) \neq 0\} \equiv \{(\hat{f}, l_{\hat{p}}) \in \overline{PD_7} : \ \Psi_{PE_6}(\hat{f}, l_{\hat{p}}) \neq 0\} \tag{1.7.31} \\
\{(\hat{f}, l_{\hat{p}}) \in \overline{\mathcal{PA}_6} : & \Psi_{PD_4}(\hat{f}, l_{\hat{p}}) = 0, \ \Psi_{PD_6}(\hat{f}, l_{\hat{p}}) = 0, \ \Psi_{PX_6}(\hat{f}, l_{\hat{p}}) \neq 0\} \equiv \{(\hat{f}, l_{\hat{p}}) \in \overline{PE_7} : \ \Psi_{PX_6}(\hat{f}, l_{\hat{p}}) \neq 0\} \tag{1.7.32} \\
\{(\hat{f}, l_{\hat{p}}) \in \overline{\mathcal{PA}_6} : & \Psi_{PD_4}(\hat{f}, l_{\hat{p}}) = 0, \ \Psi_{PE_6}(\hat{f}, l_{\hat{p}}) = 0, \ \Psi_{PX_6}(\hat{f}, l_{\hat{p}}) = 0\} \equiv \overline{\mathcal{X}^#_8}. \tag{1.7.33}
\end{align*}
\]

Observe that (1.7.31), (1.7.32) and (1.7.33) prove (1.7.30). To see this, note that

\[
\begin{align*}
\{(\hat{f}, l_{\hat{p}}) \in \overline{PE_7} : & \Psi_{PX_6}(\hat{f}, l_{\hat{p}}) = 0\} \subset \overline{\mathcal{X}^#_8} \tag{1.7.34} \\
\{(\hat{f}, l_{\hat{p}}) \in \overline{PD_7} : & \Psi_{PE_6}(\hat{f}, l_{\hat{p}}) = 0\} \subset \overline{PE_7} \cup \overline{\mathcal{X}^#_8}. \tag{1.7.35}
\end{align*}
\]

Equation (1.7.34) follows from Corollary 1.6.40 and 1.6.45. Equation (1.7.35) follows from Proposition 1.6.37 and Corollary 1.6.40 and 1.6.45. It is now easy to see that (1.7.31), (1.7.32) and (1.7.33) combined with (1.7.34) and (1.7.35) prove (1.7.30).

Let us prove (1.7.31). To see why the left hand side is a subset of the right hand side, recall (1.7.17). We also recall a result:

\[
\overline{\mathcal{X}^#_8} = \{(\hat{f}, l_{\hat{p}}) \in D \times TP^2 : \Psi_{\mathcal{A}_0}(\hat{f}, l_{\hat{p}}) = 0, \ \Psi_{\mathcal{A}_1}(\hat{f}, l_{\hat{p}}) = 0, \ \Psi_{\mathcal{D}_4}(\hat{f}, l_{\hat{p}}) = 0, \ \Psi_{\mathcal{X}_8}(\hat{f}, l_{\hat{p}}) = 0\}.
\]

Now observe that \(\overline{\mathcal{PA}_6} \subset \overline{\mathcal{PA}_5}\) is implied by Lemma 1.7.1, statement 11, whence

\[
\{(\hat{f}, l_{\hat{p}}) \in \overline{\mathcal{PA}_6} : \Psi_{PD_4}(\hat{f}, l_{\hat{p}}) = 0\} \subset \{(\hat{f}, l_{\hat{p}}) \in \overline{\mathcal{PA}_5} : \Psi_{PD_4}(\hat{f}, l_{\hat{p}}) = 0\}.
\]

By (1.7.12), the right hand side above equals \(\overline{PD_6} \cup \overline{PE_6}\). But by Lemma 1.7.1, statement 7 and statement 5, we get

\[
\overline{PD_6} \cup \overline{PE_6} = PD_6 \cup PE_6 \cup PD_7 \cup PE_7 \cup \overline{\mathcal{X}^#_8}.
\]

This implies, by (1.7.17) and (1.7.24), that

\[
\{(\hat{f}, l_{\hat{p}}) \in \overline{\mathcal{PA}_6} : \Psi_{PD_4}(\hat{f}, l_{\hat{p}}) = 0, \ \Psi_{PE_6}(\hat{f}, l_{\hat{p}}) \neq 0\} \subset \overline{PD_7}.
\]

Hence, the left hand side of (1.7.31) is a subset of its right hand side.

Next let us show that the right hand side of (1.7.31) is a subset of its left hand side. Since \(\overline{\mathcal{PA}_6}\) is a closed set, it suffices to show that

\[
\{(\hat{f}, l_{\hat{p}}) \in \overline{\mathcal{PA}_6} : \Psi_{PD_4}(\hat{f}, l_{\hat{p}}) = 0, \ \Psi_{PE_6}(\hat{f}, l_{\hat{p}}) \neq 0\} \supset \{(\hat{f}, l_{\hat{p}}) \in PD_7 : \ \Psi_{PE_6}(\hat{f}, l_{\hat{p}}) \neq 0\}. \tag{1.7.36}
\]

We will simultaneously prove (1.7.36) and also prove the following:

\[
\overline{\mathcal{PA}_7} \cap PD_7 = \emptyset. \tag{1.7.37}
\]
Claim 1.7.10. Let \((\tilde{f}, l_p) \in PD_7\). Then there exists a solution \((\tilde{f}(t), l_p(t)) \in \overline{PA}_3\) near \((\tilde{f}, l_p)\) to the set of equations

\[
\Psi_{PD_4}(\tilde{f}(t), l_p(t)) \neq 0, \Psi_{PA_4}(\tilde{f}(t), l_p(t)) = 0, \Psi_{PA_5}(\tilde{f}(t), l_p(t)) = 0, \Psi_{PA_6}(\tilde{f}(t), l_p(t)) = 0. \tag{1.7.38}
\]

Moreover, whenever such a solution \((\tilde{f}(t), l_p(t))\) is sufficiently close to \((\tilde{f}, l_p)\) it lies in \(PA_6\), i.e., \(\Psi_{PA_7}(\tilde{f}(t), l_p(t)) \neq 0\). In particular \((\tilde{f}(t), l_p(t))\) does not lie in \(PA_7\).

Note that claim 1.7.10 proves (1.7.36) and (1.7.37) simultaneously.

Proof: As before, we will first solve the equation

\[
f_{02}(t) \neq 0, \quad f_{02}(t)A_{4}^{f(t)} = 0, \quad f_{02}(t)^2A_{5}^{f(t)} = 0 \quad \text{and} \quad f_{02}(t)^3A_{6}^{f(t)} = 0. \tag{1.7.39}
\]

We claim that the only solutions to (1.7.39) that go to zero as \(f_{02}(t)\) goes to zero are of the form

\[
f_{02}(t) = u^2 + O(u^4)
\]

\[
f_{21}(t) = \frac{f_{31}(t)}{3f_{12}(t)}u^2 + \sqrt{\beta(t)}u^3 + O(u^4)
\]

\[
f_{40}(t) = \frac{f_{31}(t)^2}{3f_{12}(t)^2}u^2 + O(u^3)
\]

\[
\left(f_{50}(t) - \frac{5f_{31}(t)^2}{3f_{12}(t)}\right) = -15\beta(t)u^2 + O(u^3)
\]

where

\[
\beta(t) = -\frac{f_{02}(t)f_{31}(t)^3}{162f_{12}(t)^4} + \frac{f_{22}(t)f_{31}(t)^2}{18f_{12}(t)^3} - \frac{f_{41}(t)f_{31}(t)}{18f_{12}(t)^2} + \frac{f_{60}(t)}{90f_{12}(t)} \tag{1.7.42}
\]

for just one choice of a branch of \(\sqrt{\beta(t)}\). We will see shortly that \(\beta(t) \neq 0\). The value for \(f_{40}\) can be calculated using \(f_{21}, f_{02}\) and \(f_{02}(t)A_{4}^{f(t)} = 0\) while the fourth equation follows by using the first three equations and \(f_{02}(t)^2A_{5}^{f(t)} = 0\). Let us now explain how we obtain (1.7.40) and (1.7.41). The equation \(f_{02}(t)^3A_{6}^{f(t)} = 0\) is a cubic equation in \(f_{21}(t)\), i.e., it is of the form

\[
A_3(f_{02}(t))f_{21}(t)^3 + A_2(f_{02}(t))f_{21}(t)^2 + A_1(f_{02}(t))f_{21}(t) + A_0(f_{02}(t)) = 0.
\]

As \(\Psi_{PC_6}(\tilde{f}(t), l_p(t)) \neq 0\), this implies that \(f_{12}(t) \neq 0\). It follows that as \(f_{02}(t)\) goes to zero \(A_2\) remains non zero. Hence, there exists a unique holomorphic function \(P(f_{02}(t))\), of \(f_{02}(t)\) (close to the zero function), such that if we make a change of variables

\[
f_{21}(t) = H + P(f_{02}(t))
\]

then our cubic equation becomes

\[
\hat{A}_3(f_{02}(t))H^3 + \hat{A}_2(f_{02}(t))H^2 + \hat{A}_0(f_{02}(t)) = 0.
\]

The argument is same as in Lemma 1.4.5, where we show the existence of \(B(x)\) (it is simply an application of the Implicit Function Theorem). In fact, we observe that

\[
P(f_{02}) = \frac{1}{3A_3} \left(-A_2 + \sqrt{A_2^2 - 3A_1A_3}\right).
\]

\[\text{In other words, choosing the other branch of the square root does not give us any extra solutions.}\]
This is defined even when $A_3 = 0$ as can be seen by a standard binomial expansion, i.e.,

$$P(f_{02}(t)) = \frac{f_{31}(t)}{3f_{12}(t)}f_{02}(t) + O(f_{02}(t)^2) = \frac{f_{31}(t)}{3f_{12}(t)}u^2 + O(u^4).$$

The other root has the property that $P(f_{02}(t))$ goes to a non-zero constant as $f_{02}(t)$ goes to zero. Since $A_2(0) \neq 0$, we can divide out by $A_2(f_{20}(t))$ and get

$$\hat{A}_3(f_{02}(t))H^3 + H^2 + \hat{A}_0(f_{02}(t)) = 0. \quad (1.7.43)$$

By a simple calculation, it is easy to see that

$$\hat{A}_0(f_{02}(t)) = -\frac{\beta(t)}{f_{12}(t)}f_{02}(t)^3 + O(f_{02}(t)^4).$$

Assuming $\beta(t) \neq 0$ we can make a change of coordinates

$$\hat{f}_{02} = f_{02}(t)\left(\frac{f_{12}(t)\hat{A}_0(f_{02}(t))}{-\beta(t)f_{02}(t)^3}\right)^\frac{1}{3}, \quad \hat{H} = H(1 + \hat{A}_3(f_{02}(t))H)^\frac{1}{2}. $$

Our cubic equation (1.7.43) now becomes

$$\hat{H}^2 - \frac{\beta(t)}{f_{12}(t)}\hat{f}_{02}^3 = 0. \quad (1.7.44)$$

Now, it is easy to see that the only small solutions to (1.7.44) are of the form

$$\hat{H} = \sqrt[3]{\frac{\beta(t)}{f_{12}(t)}}u^3,$$

for just one choice of $\sqrt[3]{\beta(t)}$. In other words, by choosing just one branch of $\sqrt[3]{\beta(t)}$, we get all the possible small solutions of (1.7.44). By inverting the change of coordinates, $(H, f_{02}) \rightarrow (\hat{H}, \hat{f}_{02})$, we conclude that the only small solutions to (1.7.43) are of the form

$$H = \sqrt[3]{\frac{\beta(t)}{f_{12}(t)}}u^3 + O(u^4), \quad f_{02}(t) = u^2 + O(u^4).$$

(Note that the transformation $(H, f_{02}) \rightarrow (\hat{H}, \hat{f}_{02})$ is identity to first order, i.e. the Jacobian matrix of this transformation at the origin is the identity matrix.) Combining the expression for $P(f_{02})$ and $H$ gives us (1.7.41) and (1.7.40). It remains to show that $\beta(t) \neq 0$. To see this, note that

$$\beta(t) = \frac{D_8(t)}{90f_{12}(t)} - \frac{f_{30}(t)f_{31}(t)D_7(t)}{54f_{12}(t)^3}. \quad (1.7.45)$$

Since $(\hat{f}, l_p) \in \mathcal{P}\mathcal{D}_7$, $D_7^f = 0$ and $D_8^f \neq 0$. Therefore, by (1.7.45) $\beta(t) \neq 0$ for small $t$.

Equation (1.7.42) combined with (1.7.45) imply that the only solutions to the functional equation
The sections $(\tilde{f}(t), l_p(t))$ are transverse to the zero set (as proved in Proposition 1.6.19, 1.6.26 and 1.6.37), there exists a solution $(\tilde{f}(t), l_p(t))$ close to $(\hat{f}, \hat{l}_p)$ to (1.7.21). This proves our first assertion. Next we need to show that any such solution satisfies the condition $\Psi_{PD_7}(\tilde{f}(t), l_p(t)) \neq 0$ if $t$ is sufficiently small. To prove that we observe

\[ f_{02}(t)^4 A_6^f(t) = 630 f_{12}(t)^2 \beta(t) u^6 + O(u^7) \quad \text{using (1.7.42)} \]

\[ \Rightarrow \quad \Psi_{PD_7}(\tilde{f}(t), l_p(t)) = 630 \Psi_{\mathcal{E}_6}(\tilde{f}(t), l_p(t))^2 B(\tilde{f}(t), l_p(t)) t^6 + O(t^7) \quad (1.7.47) \]

Since $(\tilde{f}, l_p) \in PD_7$, we get that $B(\tilde{f}(t), l_p(t))^2 \neq 0$ and $\Psi_{\mathcal{E}_6}(\tilde{f}(t), l_p(t)) \neq 0$. Hence, (1.7.47) implies that if $t$ is sufficiently small then $\Psi_{PD_7}(\tilde{f}(t), l_p(t)) \neq 0$, since all the sections are continuous. This proves claim 1.7.10.

Before proving (1.7.32), let us prove a corollary which follows immediately.

**Corollary 1.7.11.** Let $W \rightarrow D \times \mathbb{P}T^2$ be a vector bundle such that the rank of $W$ is same as the dimension of $PD_7$. Let $Q : D \times \mathbb{P}T^2 \rightarrow W$ be a generic smooth section. Suppose $(\tilde{f}, l_p) \in PD_7 \cap Q^{-1}(0)$. Then the section

\[ \Psi_{PD_7} \oplus Q : \mathcal{O}_6 \rightarrow L_{PD_7} \oplus W \]

vanishes around $(\tilde{f}, l_p)$ with a multiplicity of 6.

**Proof:** Follows from the fact that $Q^{-1}(0)$ intersects $PD_7$ transversely and (1.7.47). \qed

Next we will prove (1.7.32). To show that the left hand side is a subset of the right hand side note that $\mathcal{O}_6 \subset \mathcal{O}_5$ by Lemma 1.7.1, statement 11. Consequently,

\[ \{ (\tilde{f}, l_p) \in \mathcal{O}_6 : \Psi_{PD_4}(\tilde{f}, l_p) = 0, \Psi_{\mathcal{E}_6}(\tilde{f}, l_p) = 0, \Psi_{\mathcal{X}_6}(\tilde{f}, l_p) \neq 0 \} \]

is contained in

\[ \{ (\tilde{f}, l_p) \in \mathcal{O}_5 : \Psi_{PD_4}(\tilde{f}, l_p) = 0, \Psi_{\mathcal{E}_6}(\tilde{f}, l_p) = 0, \Psi_{\mathcal{X}_6}(\tilde{f}, l_p) \neq 0 \} \].
But the quantity above, by (1.7.14), equals
\[
\{(\bar{f}, l_p) \in \overline{\mathcal{P}E}_6 : \Psi_{\mathcal{P}A_8}(\bar{f}, l_p) \neq 0\} \subset \mathcal{P}E_6 \cup \mathcal{P}E_7 \cup \overline{\mathcal{A}_8},
\]
where the last inclusion follows from Lemma 1.7.1, statement 5. Therefore, the left hand side of (1.7.32) is a subset of its right hand side, using (1.7.24).

For the converse, since \(\overline{\mathcal{P}A}_6\) is a closed set, it suffices to show that
\[
\{(\bar{f}, l_p) \in \overline{\mathcal{P}A}_6 : \Psi_{\mathcal{P}A_4}(\bar{f}, l_p) = 0, \Psi_{\mathcal{P}A_8}(\bar{f}, l_p) = 0\} \supset \{(\bar{f}, l_p) \in \mathcal{P}E_7 : \Psi_{\mathcal{P}A_8}(\bar{f}, l_p) \neq 0\}.
\]

We will simultaneously prove this statement and also prove
\[
\overline{\mathcal{P}A}_7 \cap \mathcal{P}E_7 = \emptyset. \tag{1.7.49}
\]

**Claim 1.7.12.** Let \((\bar{f}, l_p) \in \mathcal{P}E_7\). Then there exists a solution \((\bar{f}(t), l_p(t)) \in \overline{\mathcal{P}A}_3\) near \((\bar{f}, l_p)\) to the set of equations
\[
\Psi_{\mathcal{P}A_4}(\bar{f}(t), l_p(t)) \neq 0, \Psi_{\mathcal{P}A_8}(\bar{f}(t), l_p(t)) = 0, \Psi_{\mathcal{P}A_5}(\bar{f}(t), l_p(t)) = 0, \Psi_{\mathcal{P}A_6}(\bar{f}(t), l_p(t)) = 0. \tag{1.7.50}
\]
Moreover, whenever such a solution \((\bar{f}(t), l_p(t))\) is sufficiently close to \((\bar{f}, l_p)\) it lies in \(\mathcal{P}A_6\), i.e., \(\Psi_{\mathcal{P}A_5}(\bar{f}(t), l_p(t)) \neq 0\). In particular \((\bar{f}(t), l_p(t))\) does not lie in \(\mathcal{P}A_7\).

Note that claim 1.7.12 proves (1.7.48) and (1.7.49) simultaneously.

**Proof:** As before, we will first solve the equation
\[
f_{02}(t) \neq 0, \quad f_{02}(t)^2 A_4^{f(t)} = 0, \quad f_{02}(t)^2 A_5^{f(t)} = 0, \quad f_{02}(t)^3 A_6^{f(t)} = 0. \tag{1.7.51}
\]
The only solutions to (1.7.51), that converge to zero as \(f_{02}(t)\) and \(f_{12}(t)\) go to zero are
\[
f_{12}(t) = u \\
f_{21}(t) = -\frac{3}{2f_{03}(t)} u^2 + O(u^3) \\
f_{02}(t) = -\frac{9}{4f_{31}(t)f_{03}(t)} u^3 + O(u^4) \\
f_{40}(t) = -\frac{3f_{31}(t)}{f_{03}(t)} u + O(u^2) \tag{1.7.52}
\]
We set \(f_{12}(t) = u\) and use \(f_{02}(t)^2 A_5^{f(t)} = 0\) to solve for \(f_{02}/f_{21}\); we get
\[
\frac{f_{02}(t)}{f_{21}(t)} = \frac{3u}{2f_{31}(t)} + O(u^2).
\]
We now use \(f_{02}(t)^3 A_6^{f(t)} = 0\) ans replace \(f_{02}\) with the expression above to solve for \(f_{21}\) and obtain the second equation. This also implies the third equation by the expression above. The last equation can now be obtained using \(f_{02}(t)^2 A_4^{f(t)} = 0\) and the first three equations.

Let us explain the method in detail. Using \(f_{02}(t)^2 A_5^{f(t)} = 0\) we can solve for \(f_{21}/f_{02}\) and get
\[
\frac{f_{02}(t)}{f_{21}(t)} = \frac{10f_{31}(t) - \sqrt{100f_{31}(t)^2 - 60f_{50}(t)u}}{2f_{50}(t)} = \frac{3}{2f_{31}(t)} u + O(u^2) \tag{1.7.53}
\]
It is easy to see that we never really used the fact that \( f_{50}(t) \neq 0 \); the equality of the first and last term remains valid even when \( f_{50} = 0 \). However, we do need to justify why we did not choose the other branch of the square root. We will explain that shortly. Plugging in the value of \( f_{02} \) from \((1.7.53)\) in equation \( f_{02}(t)^3A_6^{f(t)} = 0 \) and by using the Implicit Function Theorem, we get the expression for \( f_{21}(t) \) in \((1.7.52)\). And now using the value of \( f_{21}(t) \) and \((1.7.53)\) we get the expression for \( f_{02}(t) \) in \((1.7.52)\).

It remains to show that why we did not chose the other branch of the square root. It is easy to see that if we chose the other branch, it would imply that as \( f_{02}(t) \) and \( f_{21}(t) \) go to zero, the ratio \( L_t := \frac{f_{21}(t)}{f_{02}(t)} \) tends to a finite number \( L \), since \( f_{31} \neq 0 \). Using \( f_{03}(t)^3A_6^{f(t)} = 0 \) we can solve for \( f_{31}(t) \) as a quadratic equation and get that

\[
 f_{31}(t) = \frac{30L_tf_{12}(t) \pm \sqrt{50} \sqrt{-15L_t^3f_{02}(t)f_{03}(t) + 45L_t^2f_{02}(t)f_{22}(t) - 15L_tf_{02}(t)f_{41}(t) + f_{02}(t)f_{60}(t)}}{10}.
\]

It is now easy to see that \( f_{31}(t) \) tends to zero as \( f_{12}(t) \) and \( f_{02}(t) \) tend to zero. This gives us a contradiction, since \( f_{31} \neq 0 \).

Since \((\tilde{f}, \tilde{l}_p) \in \mathcal{P}\mathcal{E}_7\) we get that \( f_{03}, f_{31} \neq 0 \). Equation \((1.7.52)\) implies that the only solutions to the functional equation \((1.7.50)\) are of the form

\[
 f(t, l_p(t)) = t
\]

\[
 \Psi_{\mathcal{P}\mathcal{E}_6}(\tilde{f}(t), l_p(t)) = \Psi_{\mathcal{P}\mathcal{D}_4}(\tilde{f}(t), l_p(t)) = \Psi_{\mathcal{P}\mathcal{E}_7}(\tilde{f}(t), l_p(t)) = O(t^2)
\]

where equality holds as functionals. Since the sections

\[
 \Psi_{\mathcal{P}\mathcal{D}_4} : \mathcal{P}\mathcal{A}_3 \to \mathbb{L}_{\mathcal{P}\mathcal{D}_4}, \quad \Psi_{\mathcal{P}\mathcal{D}_5} : \Psi_{\mathcal{P}\mathcal{D}_4}^{-1}(0) \to \mathbb{L}_{\mathcal{P}\mathcal{D}_5},
\]

\[
 \Psi_{\mathcal{P}\mathcal{E}_6} : \Psi_{\mathcal{P}\mathcal{D}_5}^{-1}(0) \to \mathbb{L}_{\mathcal{P}\mathcal{E}_6}, \quad \Psi_{\mathcal{P}\mathcal{E}_7} : \Psi_{\mathcal{P}\mathcal{E}_6}^{-1}(0) \to \mathbb{L}_{\mathcal{P}\mathcal{E}_7}
\]

are transverse to the zero set (as proved in Proposition 1.6.19, 1.6.26 and 1.6.31), there exists a solution \((\tilde{f}(t), l_p(t))\) close to \((\tilde{f}, l_p)\) for \((1.7.54)\). This proves our first assertion. Next we need to show that any such solution satisfies the condition \( \Psi_{\mathcal{P}\mathcal{A}_7}(\tilde{f}(t), l_p(t)) \neq 0 \) if \( t \) is sufficiently small. To prove that we observe

\[
 f_{02}(t)^4A_7^{f(t)} = \frac{-2835}{16f_{03}(t)^2}u^7 + O(u^8)
\]

using \((1.7.52)\)

\[
 \Rightarrow \quad \Psi_{\mathcal{P}\mathcal{A}_7}(\tilde{f}(t), l_p(t)) = \frac{-2835}{16\Psi_{\mathcal{P}\mathcal{X}_6}(\tilde{f}(t), l_p(t))^2}t^7 + O(t^8)
\]

Since \((\tilde{f}, l_p) \in \mathcal{P}\mathcal{E}_7\), we get that \( \Psi_{\mathcal{P}\mathcal{X}_6}(\tilde{f}, l_p) \neq 0 \). Hence, \((1.7.55)\) implies that if \( t \) is sufficiently small \( \Psi_{\mathcal{P}\mathcal{A}_7}(\tilde{f}(t), l_p(t)) \neq 0 \), since all the sections are continuous. This proves claim 1.7.12.

Before proving (1.7.33), let us prove a corollary which follows immediately.
Corollary 1.7.13. Let \( \mathbb{W} \rightarrow \mathcal{D} \times \mathbb{P}\mathbb{T}^2 \) be a vector bundle such that the rank of \( \mathbb{W} \) is same as the dimension of \( \mathcal{P}\mathcal{E}_7 \). Let \( \mathcal{Q} : \mathcal{D} \times \mathbb{P}\mathbb{T}^2 \rightarrow \mathbb{W} \) be a generic smooth section. Suppose \((\tilde{f}, l_{\tilde{p}}) \in \mathcal{P}\mathcal{E}_7 \cap \mathcal{Q}^{-1}(0) \). Then the section

\[
\Psi_{\mathcal{P}\mathcal{A}_7} + \mathcal{Q} : \overline{\mathcal{P}\mathcal{A}_5} \rightarrow \mathcal{L}\mathcal{P}\mathcal{A}_7 \oplus \mathbb{W}
\]

vanishes around \((\tilde{f}, l_{\tilde{p}})\) with a multiplicity of 7.

**Proof:** Follows from the fact that \( \mathcal{Q}^{-1}(0) \) intersects \( \mathcal{P}\mathcal{E}_7 \) transversely and \((1.7.55)\).

Finally, we will prove \((1.7.33)\). Let us show that the left hand side is contained in the right hand side. Note that \( \overline{\mathcal{P}\mathcal{A}_6} \subset \overline{\mathcal{P}\mathcal{A}_5} \) by Lemma 1.7.1, statement 11. Therefore,

\[
\{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}\mathcal{A}_6} : \Psi_{\mathcal{P}\mathcal{D}_4}(\tilde{f}, l_{\tilde{p}}) = 0, \Psi_{\mathcal{P}\mathcal{E}_6}(\tilde{f}, l_{\tilde{p}}) = 0, \Psi_{\mathcal{P}\mathcal{X}_8}(\tilde{f}, l_{\tilde{p}}) = 0\}
\]

is contained in

\[
\{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}\mathcal{A}_5} : \Psi_{\mathcal{P}\mathcal{D}_4}(\tilde{f}, l_{\tilde{p}}) = 0, \Psi_{\mathcal{P}\mathcal{E}_6}(\tilde{f}, l_{\tilde{p}}) = 0, \Psi_{\mathcal{P}\mathcal{X}_8}(\tilde{f}, l_{\tilde{p}}) = 0\}.
\]

The last quantity above equals, due to \((1.7.14)\), and \((1.7.1)\) and Corollary 1.6.45

\[
\text{LHS} = \{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}\mathcal{E}_6} : \Psi_{\mathcal{P}\mathcal{X}_8}(\tilde{f}, l_{\tilde{p}}) = 0\} = \hat{X}_8^{\#b}.
\]

This implies that the left hand side of \((1.7.33)\) is a subset of its right hand side.

Next let us show that the right hand side of \((1.7.33)\) is a subset of its left hand side. Since \( \overline{\mathcal{P}\mathcal{A}_6} \) is a closed set, it suffices to show that

\[
\{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{P}\mathcal{A}_6} : \Psi_{\mathcal{P}\mathcal{D}_4}(\tilde{f}, l_{\tilde{p}}) = 0, \Psi_{\mathcal{P}\mathcal{E}_7}(\tilde{f}, l_{\tilde{p}}) = 0, \Psi_{\mathcal{P}\mathcal{X}_8}(\tilde{f}, l_{\tilde{p}}) = 0\} \supset \hat{X}_8^{\#b}
\]

\((1.7.56)\)

We will simultaneously prove this statement and also prove

\[
\overline{\mathcal{P}\mathcal{A}_7} \cap \hat{X}_8^{\#b} = \emptyset
\]

\((1.7.57)\)

Claim 1.7.14. Let \((\tilde{f}, l_{\tilde{p}}) \in \hat{X}_8^{\#b}\). Then there exists a solution \((\hat{f}(t), l_{\hat{p}}(t)) \in \overline{\mathcal{P}\mathcal{A}_3} \) near \((\tilde{f}, l_{\tilde{p}})\) to the set of equations

\[
\Psi_{\mathcal{P}\mathcal{D}_4}(\hat{f}(t), l_{\hat{p}}(t)) \neq 0, \Psi_{\mathcal{P}\mathcal{A}_4}(\hat{f}(t), l_{\hat{p}}(t)) = 0, \Psi_{\mathcal{P}\mathcal{A}_5}(\hat{f}(t), l_{\hat{p}}(t)) = 0, \Psi_{\mathcal{P}\mathcal{A}_6}(\hat{f}(t), l_{\hat{p}}(t)) = 0.
\]

\((1.7.58)\)

Moreover, whenever such a solution \((\hat{f}(t), l_{\hat{p}}(t))\) is sufficiently close to \((\tilde{f}, l_{\tilde{p}})\) it lies in \( \mathcal{P}\mathcal{A}_6 \), i.e., \( \Psi_{\mathcal{P}\mathcal{A}_7}(\hat{f}(t), l_{\hat{p}}(t)) \neq 0 \). In particular, \((\tilde{f}(t), l_{\tilde{p}}(t))\) does not lie in \( \mathcal{P}\mathcal{A}_7 \).

Notice that claim 1.7.14 proves \((1.7.56)\) and \((1.7.57)\) simultaneously.

**Proof:** As before, we will first solve the equation

\[
f_{02}(t) \neq 0, \quad f_{02}(t)A_{4}^{f(t)} = 0, \quad f_{02}(t)^2A_{5}^{f(t)} = 0, \quad f_{02}(t)^3A_{6}^{f(t)} = 0.
\]

\((1.7.59)\)
The only solutions to (1.7.59) that converge to zero as \( f_{02}(t), f_{12}(t) \) and \( f_{03} \) go to zero are

\[
\begin{align*}
    f_{21}(t) &= u \\
    f_{02}(t) &= \frac{3u^2}{f_{40}} \quad \text{using} \quad f_{02}(t)A_4^{f(t)} = 0 \\
    f_{12}(t) &= \frac{2f_{31}}{f_{40}}u - \frac{3f_{50}}{5f_{40}^2}u^2 \quad \text{using} \quad f_{02}(t)^2A_5^{f(t)} = 0 \\
    f_{03}(t) &= \left( -\frac{6f_{31}^2}{f_{40}^2} + \frac{9f_{22}}{f_{40}} \right)u + \left( -\frac{9f_{41}}{f_{40}^2} + \frac{36f_{31}f_{50}}{5f_{40}} \right)u^2 \\
    &+ \left( -\frac{54f_{50}^2}{25f_{40}^4} + \frac{9f_{60}}{5f_{40}^3} \right)u^3 \quad \text{using} \quad f_{02}(t)^3A_6^{f(t)} = 0. \quad (1.7.60)
\end{align*}
\]

Note that since \((\tilde{f}, l_p) \in \mathcal{X}_\mathfrak{d}^{\#0}\) we get that \(f_{40} \neq 0\). Equation (1.7.60) implies that the only solutions to the functional equation (1.7.58) are of the form

\[
\begin{align*}
    \Psi_{PD_4}^{L_0}(\tilde{f}(t), l_p(t)) &= t \\
    \Psi_{PD_4}(\tilde{f}(t), l_p(t)) &= \frac{3}{\Psi_{PD_6}(\tilde{f}(t), l_p(t))}t^2 \\
    \Psi_{PE_6}(\tilde{f}(t), l_p(t)) &= \frac{2\Psi_{PE_6}(\tilde{f}(t), l_p(t))}{\Psi_{PD_6}(\tilde{f}(t), l_p(t))}t + O(t^2) \\
    \Psi_{PX_8}(\tilde{f}(t), l_p(t)) &= \left( -\frac{6\Psi_{PE_6}(\tilde{f}(t), l_p(t))^2}{\Psi_{PD_6}(\tilde{f}(t), l_p(t))^2} + \frac{9\varphi(\tilde{f}(t), l_p(t))}{\Psi_{PE_7}(\tilde{f}(t), l_p(t))} \right)t + O(t^2). \quad (1.7.61)
\end{align*}
\]

The functional \(\varphi\) is given by

\[
\{\varphi(\tilde{f}(t), l_p(t))\} = \{f \otimes p^{\otimes d} \otimes w^{\otimes 2} \otimes \tilde{w}^{\otimes 2}\} := f_{22},
\]

where notations are as defined in subsection 1.3.2. Equality holds here as functionals. Since the sections

\[
\begin{align*}
    \Psi_{PD_4} : \overline{\mathcal{A}}_3 \rightarrow \mathbb{L}_{PD_4}, \quad \Psi_{PD_5} : \mathbb{L}_{PD_5}(0) \rightarrow \mathbb{L}_{PD_5}, \\
    \Psi_{PE_6} : \mathbb{L}_{PE_6}(0) \rightarrow \mathbb{L}_{PE_6}, \quad \Psi_{PX_8} : \mathbb{L}_{PX_8}(0) \rightarrow \mathbb{L}_{PX_8}
\end{align*}
\]

are transverse to the zero set (as proved in Proposition 1.6.19, 1.6.26 and 1.6.31), there exists a solution \((\tilde{f}(t), l_p(t))\) close to \((\tilde{f}, l_p)\) to (1.7.61). This proves our first assertion. Next we need to show that any such solution satisfies the condition \(\Psi_{PD_7}(\tilde{f}(t), l_p(t)) \neq 0\) if \(t\) is small. Observe

\[
\begin{align*}
    f_{02}(t)^4A_7^{f(t)} &= \left( -\frac{f_{31}(t)^3}{8f_{40}(t)^3} + \frac{3f_{22}(t)f_{31}(t)}{16f_{40}(t)^2} - \frac{f_{13}(t)}{16f_{40}(t)} \right)u^5 + O(u^6) \quad \text{using} \quad (1.7.60) \\
    \Rightarrow \Psi_{PD_7}(\tilde{f}(t), l_p(t)) &= \left( \frac{\Psi_{PD_7}(\tilde{f}(t), l_p(t))}{\Psi_{PD_7}(\tilde{f}(t), l_p(t))^3} \right)u^5 + O(u^6) \quad (1.7.62)
\end{align*}
\]

Since \((\tilde{f}, l_p) \in \mathcal{X}_\mathfrak{d}^{\#0}\), we get that \(\Psi_{PD_7}(\tilde{f}, l_p) \neq 0\) and \(\Psi_{PD_7}(\tilde{f}, l_p) \neq 0\). Hence, (1.7.62) implies that if \(t\) is small \(\Psi_{PD_7}(\tilde{f}(t), l_p(t)) \neq 0\), which proves claim 1.7.14. This finishes the proof of Lemma 1.7.1, statement 12. \(\square\)
1.8 Low degree checks

Verification of the number $N(A_1, 0) = 3(d - 1)^2$:
- $d = 1$: There are no nodal lines.
- $d = 2$: The number of line pairs that pass through 4 general points is \( \frac{1}{2} \binom{4}{2} = 3 \).
- $d = 3$: The number of nodal cubics passing through 8 general points are the rational cubics passing through these points; this number 12 can also be computed through Kontsevich’s recursion formula.

Verification of the number $N(A_1, 1) = 3d - 1$:
- $d = 1$: There are no nodal lines.
- $d = 2$: The number of line pairs that pass through 3 points and meet on a line is \( \binom{3}{2} = 3 \).

Verification of the number $N(A_2, 0) = 12(d - 1)(d - 2)$:
- $d = 1$: There are no lines with a cusp.
- $d = 2$: The only way a conic can have a cusp is if it’s a double line. There are no double lines through three generic points.
- $d = 3$: The number of nodal cubics passing through 7 general points are the rational cubics passing through these points; according to [3] this number 24. This number can also be computed by the algorithm described in [5].
- $d = 4$: The number of quartics with a cusp is 72. This is same as the number of genus two curves with a cusp and equals 72 (cf. [4], pp. 19).

Verification of the number $N(A_4, 0) = 60(d - 3)(3d - 5)$:
- $d = 3$: There are no cubics with an $A_4$-node.

Verification of the number $N(D_4, 0) = 15(d - 2)^2$:
- $d = 2$: There are no conics with a $D_4$-node.
- $d = 3$: The only way a cubic can have a $D_4$-node is, if it breaks into three distinct lines intersecting at a common point. The number of such configurations passing through 5 points is \( \frac{1}{3} \times \binom{5}{2} \times \binom{3}{2} = 15 \).

Verification of the number $N(D_4, 1) = 6(d - 2)$:
- $d = 2$: There are no conics with a $D_4$-node on a line.
- $d = 3$: The number of triple lines, having a common point at a given line and passing through four points is \( \binom{4}{2} = 6 \).

Verification of the number $N(E_6, 0) = 21(d - 3)(4d - 9)$:
- $d = 3$: There are no cubics with an $E_6$-node.
- $d = 4$: An $E_6$-node contributes three to the genus of a curve. Since a smooth quartic has genus three, the quartics with an $E_6$-node have genus zero. The number of such quartics through 8 points is 147 (cf. [6], pp. 24).

1.8.1 Verification of the number $N(PA_3, n, m)$

In this subsection we will show that we can verify the number $N(PA_3, n, m)$ for $d = 3$ for all values of $n$ and $m$. Note that a cubic will have an $A_3$-node if and only if the cubic breaks into a conic and a line that is tangent to the conic. Hence, let us consider the following spaces and line
bundles:
\[ \mathbb{P}^2 : \text{two dimensional complex projective space }, \quad \gamma_{\mathbb{P}^2} \longrightarrow \mathbb{P}^2 \]
\[ D_2 : \text{space of conics in } \mathbb{P}^2 \cong \mathbb{P}^5, \quad \gamma_{D_2} \longrightarrow D_2 \]
\[ D_1 : \text{space of lines in } \mathbb{P}^2 \cong \mathbb{P}^2, \quad \gamma_{D_1} \longrightarrow D_1 \]
\[ Z := \{(\tilde{f}_1, \tilde{p}) \in D_1 \times \mathbb{P}^2 : f_1(p) = 0\}. \]

We would like to consider the space of conics and a line, such that the line is tangent to the conic. Describing this space requires a little bit of thought. First, let us take a simple example to see what is going on. Consider the conic \( f_2 \) and line \( f_1 \) given by
\[
 f_2(X, Y, Z) = YZ - X^2, \quad f_1(X, Y, Z) = Y
\]
Let us look at the neighbourhood of the point \([0 : 0 : 1]\). The line \( f_1^{-1}(0) \) is tangent to the conic \( f_2^{-1}(0) \) at \([0 : 0 : 1]\). Let us look at this more closely. In affine coordinates, the line \( f_1^{-1}(0) \) is basically the \( x \)-axis \((y = 0)\). The fact that this line is tangent to the conic, is equivalent to the statement
\[
 \frac{\partial(y - x^2)}{\partial x} \bigg|_{(0,0)} = 0.
\]
Geometrically, the tangent vector \( \partial_x \) belongs to the kernel of \( \nabla f_1 \) at the point \([0 : 0 : 1]\). Hence, what we have is that the line \( f_1^{-1}(0) \) is tangent to the conic \( f_2^{-1}(0) \) at the point \( \tilde{p} \), if and only if \( \nabla f_2|_{\tilde{p}} \) vanishes, restricted to the Kernel of \( \nabla f_1|_{\tilde{p}} \), i.e.
\[
 \nabla f_2|_{\tilde{p}}(u) = 0 \quad \forall u \in \text{Ker}(\nabla f_1|_{\tilde{p}}).
\]

Given a line \( \tilde{f}_1 \in D_1 \) and a point \( \tilde{p} \in \mathbb{P}^2 \) on the line, we obtain a short exact sequence of vector bundles over \( Z \):
\[
 0 \longrightarrow \text{Ker}(\nabla f_1|_{\tilde{p}}) \longrightarrow T\mathbb{P}^2|_{\tilde{p}} \longrightarrow \gamma_{D_1}^* \otimes \gamma_{\mathbb{P}^2}^* \longrightarrow 0. \tag{1.8.1}
\]
Let \( L := \text{Ker}(\nabla f_1|_{\tilde{p}}) \rightarrow Z \). Consider the following space:
\[
 \mathcal{T} := \{(\tilde{f}_1, \tilde{f}_2, \tilde{p}) \in D_1 \times D_2 \times \mathbb{P}^2 : f_1(p) = 0, \ f_2(p) = 0, \ \nabla f_2|_{\tilde{p}}(u) = 0 \quad \forall u \in L\}.
\]
After interpreting \( \mathcal{T} \) appropriately as the zero set of a vector bundle we can see that
\[
 \text{PD}_M(\mathcal{T}) = e(\gamma_{D_1}^* \otimes \gamma_{\mathbb{P}^2}^*)e(\gamma_{D_2}^* \otimes \gamma_{\mathbb{P}^2}^*)e(\gamma_{D_2}^* \otimes \gamma_{\mathbb{P}^2}^* \otimes L^*) \quad \text{where } M := D_1 \times D_2 \times \mathbb{P}^2.
\]
It is now easy to see that \( \mathcal{T} \) is the space of conics and a line, such that the line is tangent to the conic. We also note that \( L \) is the distinguished direction along which the Hessian of the cubic is degenerate (one can take the specific example we have stated to see clearly what is going on). We are now ready to compute the numbers \( N(\mathcal{P}, \mathcal{A}_3, n, m) \) for \( d = 3 \).

Let us denote the first Chern classes of \( \gamma_{D_1}^* \), \( \gamma_{D_2}^* \), \( \gamma_{\mathbb{P}^2}^* \) and \( L^* \) as \( y_1, y_2, a \) and \( \lambda \) respectively. By the short exact sequence \( (1.8.1) \), we get that
\[
 \lambda = y_1 - 2a.
\]
Note that dimension of \( M \) is 9. Let
\[
 A_{n,m,k} = \langle y_2^{6-(n+m+k)} y_1^k a^n \lambda^m \text{PD}_M(\mathcal{T}), \ [D_1 \times D_2 \times \mathbb{P}^2]\rangle
 = \text{Coefficient of } y_2^5 y_1^2 a^2 \text{ in } y_2^{6-(n+m+k)} y_1^k a^n \lambda^m \text{PD}_M(\mathcal{T})
\]
Suppose we wish to compute $N(PA_3,0,0)$. This is a codimension 3 number. A generic cubic can pass through 9 points. Hence, this cubic passes through 6 points. We wish to compute in how many different ways the cubic can split as a conic and a line tangent to that conic and also passing through those 6 points. There are three possibilities: first of all we can place a conic through the 6 points and place the remaining line tangent to the conic. Obviously this number is zero, since a conic will not pass through 6 points. In addition, a little bit of thought will reveal that this number also happens to be $A_{0,0,0}$ (which is indeed 0, as expected geometrically). Next, we can choose any of the five points and place a conic through them and then we can count the number of lines through the remaining point point tangent to a conic. A little bit of thought will reveal that this number is $A_{0,0,1}$. Finally, we can place the line through 2 points and count how many conics are there through 4 points tangent to that line. A little bit of thought will reveal that this number is $A_{0,0,2}$. Of course, we can keep continuing like this all the way, but all the subsequent numbers will be zero, since a line can not pass through three or more points. Hence, what we expect is that for $d = 3$,

$$N(PA_3,0,0) = \binom{6}{6}A_{0,0,0} + \binom{6}{5}A_{0,0,1} + \binom{6}{4}A_{0,0,2} = \sum_{k=0}^{6} \binom{6}{6-k}A_{0,0,k} = 42.$$  

This agrees with our formula for general $d$.

A similar argument now shows that for all $m$ and $n$, and for $d = 3$

$$N(PA_3,n,m) = \sum_{k=0}^{6-(n+m)} \left( \frac{6-(n+m)}{6-(n+m+k)} \right) A_{n,m,k}.$$  

A simple calculation for each value of $m$ and $n$ shows that we recover the answer we obtain in [1].

### 1.8.2 Verification of the number $N(PD_6, n, m)$

In this subsection we will verify the number $N(PD_6, n, m)$ for $d = 4$ for all values of $n$ and $m$. First of all note that a quartic will have a $D_6$-node, if and only if the quartic breaks into a nodal cubic and a line, such that second derivative of the cubic along the line is zero. One can take take a simple example to see what is going on: for example look at the curve $(y - x^2)xy = 0$. The curve $(y - x^2)x = 0$ is the nodal cubic and $y = 0$ is the line. Let us now consider the following spaces and line bundles,

- $\mathbb{P}^2 :=$ projective space, $\gamma_{\mathbb{P}^2} \rightarrow \mathbb{P}^2$,
- $D_3 :=$ space of cubics in $\mathbb{P}^2 \approx \mathbb{P}^5$, $\gamma_{D_3} \rightarrow D_3$,
- $D_1 :=$ space of lines in $\mathbb{P}^2 \approx \mathbb{P}^2$, $\gamma_{D_1} \rightarrow D_1$
- $\mathcal{Z} := \{(\bar{\mathcal{f}_1}, \bar{p}) \in D_1 \times \mathbb{P}^2 : f_1(p) = 0\}$.

As before, given a line $\mathcal{f}_1 \in D_1$ and a point $\bar{p} \in \mathbb{P}^2$ on the line, we obtain a short exact sequence of vector bundles over $\mathcal{Z}$

$$0 \longrightarrow \text{Ker}(\nabla f_1|_{\bar{p}}) \longrightarrow T_{\mathbb{P}^2}|_{\bar{p}} \longrightarrow \mathbb{P}^2|_{\bar{p}} \longrightarrow 0$$

Now, let us consider the following space;

$$\mathcal{T} := \{(\mathcal{f}_1, \mathcal{f}_3, \bar{p}) \in D_1 \times D_3 \times \mathbb{P}^2 : f_1(p) = 0, \ f_3(p) = 0, \ \nabla f_3|p = 0, \ \nabla^2 f_3|p(u, u) = 0 \ \forall u \in \mathbb{L}\}.$$
After interpreting $\mathcal{T}$ appropriately as the zero set of a vector bundle, we can see that

$$\text{PD}_M(\mathcal{T}) = e(\gamma^*_D \otimes \gamma^*_P)e(\gamma^*_D \otimes \gamma^*_P^2)e(\gamma^*_D \otimes \gamma^*_P^3 \otimes T^*\mathbb{P}^2)e(\gamma^*_D \otimes \gamma^*_P^3 \otimes \mathbb{L}^*\mathbb{P}^2)$$

where $M := \mathcal{D}_1 \times \mathcal{D}_3 \times \mathbb{P}^2$.

It is easy to see that $\mathcal{T}$ is the space of nodal cubics and a line, such that second derivative of the cubic along the line is zero. We also note that $\mathbb{L}$ is the distinguished direction along which the third derivative tensor of the quartic is degenerate, i.e. $\nabla^3 (f_1 f_3)|_p(u, u, \cdot) = 0$ for all $u \in \mathbb{L}$. We are now ready to compute the numbers $\mathcal{N}(\mathcal{PD}_6, n, m)$ for $d = 4$.

Let us denote the first Chern classes of $\gamma^*_D$, $\gamma^*_P$, and $\mathbb{L}$ as $y_1$, $y_3$, $a$, and $\lambda$ respectively. By the short exact sequence of (1.8.2), we get that

$$\lambda = y_1 - 2a.$$ 

Note that dimension of $M$ is 13. Let

$$A_{n,m,k} := \langle y_2^{9-(n+m+k)} y_1^k a^n \lambda^m \text{PD}_M(\mathcal{T}), [M]\rangle$$

$$= \text{Coefficient of } y_2^9 y_1^k a^n \lambda^m \text{PD}_M(\mathcal{T})$$

Suppose we wish to compute $\mathcal{N}(\mathcal{PD}_6, 0, 0)$. This is a codimension 6 number. A generic quartic can pass through 14 points. Hence, this quartic passes through 8 points. We wish to compute in how many different ways the quartic passing through eight points can split into a nodal cubic and a line such that the second derivative of the cubic along the line is zero. There are three possibilities: first of all we can place the cubic through the 8 points and place the remaining line through the cubic in the appropriate way. A little bit of thought will reveal that this number is $A_{0,0,0}$. Next, we can choose any of the 7 points and place a nodal cubic through them and then we can count the number of lines through the remaining point that intersects the cubic in the appropriate way. A little bit of thought will reveal that this number is $A_{0,0,1}$. Finally, we can place the line through 2 points and count how many nodal cubics are there through 6 points intersecting the line in the appropriate way. A little bit of thought will reveal that this number is $A_{0,0,2}$. Of course, we can keep continuing like this all the way, but all the subsequent numbers will be zero, since a line can not pass through three or more points. Hence, what we expect is that for $d = 4$,

$$\mathcal{N}(\mathcal{PD}_6, 0, 0) = \binom{8}{8} A_{0,0,0} + \binom{8}{7} A_{0,0,1} + \binom{8}{6} A_{0,0,2} = \sum_{k=0}^{8} \binom{8}{8-k} A_{0,0,k} = 308.$$

This agrees with our formula for general $d$.

A similar argument now shows that for all $m$ and $n$, and $d = 4$:

$$\mathcal{N}(\mathcal{PD}_6, n, m) = \sum_{k=0}^{8-(n+m)} \binom{8-(n+m)}{8-(n+m+k)} A_{n,m,k}.$$ 

A straightforward calculation for each value of $m$ and $n$ now shows that we recover the answer we obtain in [1].

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1.8.3 Verification of the number $N(\mathcal{PD}_4, 0, 1)$

In this subsection we will show that we can verify the number $N(\mathcal{PD}_4, 0, 1)$ for $d = 3$. First we note that a cubic has a $D_4$ node, if and only if it breaks into three distinct lines. Let us start with the verification of $N(\mathcal{PD}_4, 0, 1)$. This is a codimension 5 number. A general cubic passes through 9 points. Hence, this cubic passes through 4 points. Given 4 generic points, we define two subspaces and $T_1$ and $T_2$ as follows:

(1) First choose any two of the four points and place a line through them. Through the remaining two points, there is a unique line and these two lines intersect at some point $\tilde{p}$. Let $T_1$ be the space of lines through $\tilde{p}$. Collectively, these three lines give us a cubic with a triple point.

(2) Now choose any two of the four points and place a line $l$ through them. Now, consider the space of the pair of lines through the remaining two points that intersect at a common point in $l$. Denote this space to be $T_2$.

Next, let $D_3$ be the space of cubics. The spaces $T_1$ and $T_2$ can be thought of as subspaces of $D_3 \times \mathbb{P}^2$. Finally, let $\pi : D_3 \times \mathbb{P}^2 \to \mathbb{P}^2$ and $\tilde{T}_i := \pi^{-1}(T_i)$. Let us now define the following tautological line bundles and the corresponding Chern classes of their dual:

$$
\begin{align*}
\gamma_D & \to D_3, & y = c_1(\gamma_D^*), \\
\gamma_{\mathbb{P}^2} & \to \mathbb{P}^2, & a = c_1(\gamma_{\mathbb{P}^2}^*), \\
\gamma & \to \mathbb{P}T\mathbb{P}^2, & \lambda = c_1(\gamma^*).
\end{align*}
$$

We make the usual abuse of notation of omitting the pullback by projection maps for vector bundles and cohomology classes. We now claim the following:

$$
\begin{align*}
\langle y, [T_1] \rangle &= 1, & \langle a, [T_1] \rangle &= 0, & \langle \lambda y, [\tilde{T}_1] \rangle &= 1, & \langle \lambda a, [\tilde{T}_1] \rangle &= 0 \\
\langle y, [T_2] \rangle &= 2, & \langle a, [T_2] \rangle &= 1, & \langle \lambda y, [\tilde{T}_2] \rangle &= 2, & \langle \lambda a, [\tilde{T}_2] \rangle &= 1.
\end{align*}
$$

It is easy to see these equalities geometrically. For example, let us see the significance of the number $\langle y, [T_1] \rangle$. This is the number of lines through two points, which is obviously one. Similarly, $\langle a, [T_1] \rangle$ is the number of lines passing through the intersection of a given point and a generic line, which is obviously zero. We can interpret the other numbers geometrically as well. Next we note that

$$
\lambda^2 = -3a\lambda - 3a^2.
$$

Using all of this, we get

$$
\begin{align*}
\langle \lambda(3\lambda + y + 3a), [\tilde{T}_1] \rangle &= 1 \\
\langle \lambda(3\lambda + y + 3a), [\tilde{T}_2] \rangle &= -4
\end{align*}
$$

There are a total of 3 ways to place two identical lines through four points. There are a total of 6 ways to place a line through any two points out of four given points. Hence, the total value is

$$
N(\mathcal{PD}_4, 0, 1) = 3 \times 1 + 6 \times (-4) = -21 \quad \text{for } d = 3.
$$

This agrees with our answer.
1.8.4 Verification of the number $\mathcal{N}(\mathcal{PD}_4, 1, 1)$

In this subsection we will show that we can verify the number $\mathcal{N}(\mathcal{PD}_4, 1, 1)$ for $d = 3$. Again, we recall that a cubic has a $\mathcal{D}_4$ node, if and only if it breaks into three distinct lines. Let us now verify $\mathcal{N}(\mathcal{PD}_4, 1, 1)$. This is a codimension 6 number. The cubic is going to pass through 3 points and a given line. We now define two spaces $\mathcal{T}_1$ and $\mathcal{T}_2$:

1. First choose any two of the three points and place a line. This line intersects the given line at some point $\tilde{p}$. Join the remaining given point with $\tilde{p}$. And now, let $\mathcal{T}_1$ be the space of lines through this point $\tilde{p}$.

2. Next, consider the space of three lines through the three given points that have a common intersection point that lies on the given line. Let us denote this space to be $\mathcal{T}_2$.

It is easy to see geometrically that

$$\langle y, [\mathcal{T}_1] \rangle = 1, \quad \langle a, [\mathcal{T}_1] \rangle = 0, \quad \langle \lambda y, [\hat{\mathcal{T}}_1] \rangle = 1, \quad \langle \lambda a, [\hat{\mathcal{T}}_1] \rangle = 0$$

$$\langle y, [\mathcal{T}_2] \rangle = 3, \quad \langle a, [\mathcal{T}_2] \rangle = -1, \quad \langle \lambda y, [\hat{\mathcal{T}}_2] \rangle = 3, \quad \langle \lambda a, [\hat{\mathcal{T}}_2] \rangle = -1.$$

Here $y$, $a$ and $\lambda$ are defined similarly as in (1.8.2). Next we note that

$$\lambda^2 = -3a\lambda - 3a^2.$$

Using all this, we get that

$$\langle \lambda(3\lambda + y + 3a), [\hat{\mathcal{T}}_1] \rangle = 1$$
$$\langle \lambda(3\lambda + y + 3a), [\hat{\mathcal{T}}_2] \rangle = -3$$

There are a total of 3 ways to choose two points from three given points. And, there is only one way to chose three points from three given points. Hence, the total value is

$$\mathcal{N}(\mathcal{PD}_4, 1, 1) = 3 \times 1 - 1 \times 3 = 0 \quad \text{for } d = 3.$$

This agrees with our answer.

1.8.5 Verification of the number $\mathcal{N}(\mathcal{PD}_4, 2, 1)$

In this subsection we will show that we can verify the number $\mathcal{N}(\mathcal{PD}_4, 2, 1)$ for $d = 3$. Again, we recall that a cubic has a $\mathcal{D}_4$ node, if and only if it breaks into three distinct lines. Let us now verify $\mathcal{N}(\mathcal{PD}_4, 2, 1)$. This is a codimension 7 number. A general cubic passes through 9 points. This cubic is going to pass through 2 points and another given point $\tilde{p}$ (which is the intersection of two given lines).

We now define the space $\mathcal{T}_1$ as follows: First join the two given points to the third point $\tilde{p}$. And now consider the space of lines through this point $\tilde{p}$. Let us denote this space to be $\mathcal{T}_1$. It is easy to see geometrically that

$$\langle y, [\mathcal{T}_1] \rangle = 1, \quad \langle a, [\mathcal{T}_1] \rangle = 0, \quad \langle \lambda y, [\hat{\mathcal{T}}_1] \rangle = 1, \quad \langle \lambda a, [\hat{\mathcal{T}}_1] \rangle = 0.$$

Again, $y$, $a$ and $\lambda$ are defined similarly as in (1.8.2). Next we note that

$$\lambda^2 = -3a\lambda - 3a^2.$$
Combining all of this we get that
\[
\langle \lambda(3\lambda + y + 3a), \ [\hat{T}_1] \rangle = 1
\]

There is only 1 way to choose two points from two given points. Hence, the total value is
\[
N(\mathcal{PD}_4, 2, 1) = 1 \times 1 = 1 \quad \text{for } d = 3.
\]

This agrees with our answer in [1].
Chapter 2

Details omitted in [2]

2.1 Review of notation

Let us recapitulate the notation that was introduced in [1] and [2].

We denote the space of degree $d$-curves in $\mathbb{P}^2$ by $\mathcal{D}$. It follows that $\mathcal{D} \cong \mathbb{P}^{\delta_d}$, where $\delta_d = d(d+3)/2$. Let $\gamma_{\mathbb{P}^2} \rightarrow \mathbb{P}^2$ be the tautological line bundle. A homogeneous degree $d$-polynomial $f$ (in 3 variables) induces a holomorphic section of the line bundle $\gamma_{\mathbb{P}^2}^d \rightarrow \mathbb{P}^2$. If $f$ is non-zero, then we will denote its equivalence class in $\mathcal{D}$ by $\tilde{f}$. Similarly, if $p$ is a non-zero vector in $\mathbb{C}^2$, we will denote its equivalence class in $\mathbb{P}^2$ by $\tilde{p}$.

Given a singularity of type $\mathcal{X}_k$, we also denote the space of degree $d$-curves with a marked point $\tilde{p}$ such that the curve has a singularity of type $\mathcal{X}_k$ at $\tilde{p}$, by the symbol $\mathcal{X}_k$. Furthermore, let us denote $A_1 \circ \mathcal{X}_k$ to be the space of degree $d$-curves with two distinct marked points $\tilde{p}_1$ and $\tilde{p}_2$ such that the curve has a an $A_1$-node at $\tilde{p}_1$ and a singularity of type $\mathcal{X}_k$ at $\tilde{p}_2$. Hence

\[ \mathcal{X}_k := \{(\tilde{f}, \tilde{p}) \in \mathcal{D} \times \mathbb{P}^2 : f\text{ has a singularity of type } \mathcal{X}_k\text{ at the point } \tilde{p}\}, \]

\[ A_1 \circ \mathcal{X}_k := \{(\tilde{f}, \tilde{p}_1, \tilde{p}_2) \in \mathcal{D} \times \mathbb{P}^2 \times \mathbb{P}^2 : f\text{ has a node at } \tilde{p}_1, \text{ has a singularity of type } \mathcal{X}_k \text{ at the point } \tilde{p}_2 \text{ and } \tilde{p}_1 \neq \tilde{p}_2\}. \]

Finally, let $\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_{\delta_d-(k+n+1)}$ be $\delta_d-(k+n+1)$ generic points in $\mathbb{P}^2$ and $L_1, L_2, \ldots, L_n$ be $n$ generic lines in $\mathbb{P}^2$. Define the following sets

\[ H_i := \{ \tilde{f} \in \mathcal{D} : f(p_i) = 0 \}, \quad H_i^* := \{ \tilde{f} \in \mathcal{D} : f(p_i) = 0, \nabla f|_{p_i} \neq 0 \}, \]

\[ \hat{H}_i := H_i \times \mathbb{P}^2 \times \mathbb{P}^2, \quad \hat{H}_i^* := H_i^* \times \mathbb{P}^2 \times \mathbb{P}^2 \text{ and } \hat{L}_i := \mathcal{D} \times \mathbb{P}^2 \times L_i. \]  

(2.1.1)

2.2 General position argument

Here we give proof of Lemma 2.7 in [2]; we restate it here with a precise bound on $d$.

**Lemma 2.2.1. (cf. Lemma 2.7, [2])** Let $\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_{\delta_d-(k+n)}$ be $\delta_d-(k+n+1)$ generic points in $\mathbb{P}^2$ and $L_1, L_2, \ldots, L_n$ be $n$ generic lines in $\mathbb{P}^2$. Let $\hat{H}_i$, $\hat{H}_i^*$ and $\hat{L}_i$ be as defined in (2.1.1). Then

\[ \overline{A_1 \circ \mathcal{X}_k} \cap \hat{H}_1 \cap \ldots \cap \hat{H}_{\delta_d-(k+n+1)} \cap \hat{L}_1 \cap \ldots \cap \hat{L}_n = \mathcal{X}_k \cap \hat{H}_1^* \cap \ldots \cap \hat{H}_{\delta_d-(k+n)}^* \cap \hat{L}_1 \cap \ldots \cap \hat{L}_n \]

\footnote{In [1] and [2] we use the symbol $\hat{A}$ to denote the equivalence class of $A$ instead of the standard $[A]$, which makes some of the calculations easier to read.}
and every intersection is transverse, provided \( d \geq \mathcal{C}_{A_1, \mathcal{X}_k} \) and \( k \geq 1 \).

**Remark 2.2.2.** We recall that

\[
\mathcal{C}_{A_1, \mathcal{A}_k} = k + 3, \quad \mathcal{C}_{A_1, \mathcal{D}_k} = k + 1, \quad \mathcal{C}_{A_1, \mathcal{E}_6} = 6, \quad \mathcal{C}_{A_1, \mathcal{E}_7} = 6.
\]

It suffices to prove this for the case \( n = 0 \). The proofs for \( n = 1 \) and \( n = 2 \) are identical. For \( n > 2 \) the Lemma is trivially true, since both the sets are empty.

**Proof:** The proof is practically same as the proof of Lemma 2.6 in [1]; the only change will occur in Lemma 1.1.14. We will now state and prove the modified Lemma that will be needed.

First, let us introduce a couple of new notations. Given a singularity of type \( \mathcal{X}_k \) and a point \( \tilde{p} \in \mathbb{P}^2 \), we define

\[
\mathcal{A}_1 \circ \mathcal{X}_k(\tilde{p}) := \{(f, \tilde{q}) \in D \times \mathbb{P}^2 : f \text{ has a singularity of type } \mathcal{A}_1 \text{ at } \tilde{q} \text{ and } \tilde{p} \text{ respectively and } \tilde{q} \neq \tilde{p}\},
\]

\[
\mathcal{A}_1 \circ \mathcal{X}_k^*(\tilde{p}) := \{(f, \tilde{q}) \in D \times \mathbb{P}^2 : f \text{ has a singularity of type } \mathcal{A}_1 \text{ at } \tilde{q} \text{ and } \tilde{p} \text{ respectively,} \quad \tilde{q} \neq \tilde{p} \text{ and } f \text{ has no other singular points}\}.
\]

**Lemma 2.2.3.** The space \( \mathcal{A}_1 \circ \mathcal{X}_k^*(\tilde{p}) \) is open and dense in \( \mathcal{A}_1 \circ \mathcal{X}_k(\tilde{p}) \) for all \( \tilde{p} \) if \( d \geq \mathcal{C}_{A_1, \mathcal{X}_k} \).

**Proof:** It is equivalent to showing that \( \mathcal{A}_1 \circ \mathcal{X}_k^*(\tilde{p}) \) is open and dense inside \( \mathcal{A}_1 \circ \mathcal{X}_k(\tilde{p}) \), the closure of \( \mathcal{A}_1 \circ \mathcal{X}_k(\tilde{p}) \) inside \( D \times \mathbb{P}^2 \). The family of curves that belongs to \( \mathcal{A}_1 \circ \mathcal{X}_k(\tilde{p}) \) forms a linear system. We claim that the base locus of this family is \( \tilde{p} \). Assuming this, we are done, because by Bertini’s theorem a generic element of \( \mathcal{A}_1 \circ \mathcal{X}_k(\tilde{p}) \) is smooth away from the base locus. Hence, \( \mathcal{A}_1 \circ \mathcal{X}_k^*(\tilde{p}) \) is open and dense inside \( \mathcal{A}_1 \circ \mathcal{X}_k(\tilde{p}) \).

In order to prove that the base locus is \( \tilde{p} \), let us first assume \( \mathcal{X}_k = \mathcal{A}_k \). Without loss of generality, we can assume that \( \tilde{p} = [0 : 0 : 1] \). Consider the family of curves

\[
(y^2 z^{d-2} + x^k z^{d-(k+1)})((x - z)^2 + y^2) = 0,
\]

\[
(y^2 z^{d-2} + x^k z^{d-(k+1)} + x^d)((x - z)^2 + (y - z)^2) = 0,
\]

\[
(y^2 z^{d-2} + x^k z^{d-(k+1)} + y^d)((x - z)^2 + (y - z)^2) = 0.
\]

These have an \( \mathcal{A}_k \)-node at \([0 : 0 : 1]\) and an \( \mathcal{A}_1 \)-node at another point. Their common zero is \([0 : 0 : 1]\). This argument works, provided \( d \geq \max(4, k + 3) = \mathcal{C}_{A_1, \mathcal{A}_k} \), which proves the claim for \( \mathcal{A}_k \).

Now let \( \mathcal{X}_k = \mathcal{D}_k \). Consider the family of curves

\[
(xy^2 z^{d-3} + x^k z^{d-(k-1)})((x - z)^2 + y^2) = 0,
\]

\[
(xy^2 z^{d-3} + x^k z^{d-(k-1)} + x^d)((x - z)^2 + (y - z)^2) = 0,
\]

\[
(xy^2 z^{d-3} + x^k z^{d-(k-1)} + y^d)((x - z)^2 + (y - z)^2) = 0.
\]

It is obvious that the first and second curves have a \( \mathcal{D}_k \)-node at \([0 : 0 : 1]\). To see why the third curve has a \( \mathcal{D}_k \)-node, we can simply use Lemma 4.6 in [1]. Hence, these curves have a \( \mathcal{D}_k \)-node at \([0 : 0 : 1]\) and an \( \mathcal{A}_1 \)-node at another point. The base locus of this family is \([0 : 0 : 1]\). This argument works, provided \( d \geq \max(5, k + 1) = \mathcal{C}_{A_1, \mathcal{D}_k} \), which proves the claim for \( \mathcal{D}_k \).

For \( \mathcal{E}_6, \mathcal{E}_7 \) we can consider the family of curves

\[
(y^3 z^{d-3} + x^4 z^{d-4})((x - z)^2 + y^2) = 0,
\]

\[
(y^3 z^{d-3} + x^4 z^{d-4} + x^d)((x - z)^2 + (y - z)^2) = 0,
\]

\[
(y^3 z^{d-3} + x^4 z^{d-4} + y^d)((x - z)^2 + (y - z)^2) = 0
\]

(2.2.4)
and

\[
(y^3 z^d - 3 + y x^3 z^{d-4})(x - z)^2 + y^2) = 0,
\]

\[
(y^3 z^d - 3 + y x^3 z^{d-4} + y^4)(x^2 + (y - z)^2) = 0,
\]

\[
(y^3 z^d - 3 + y x^3 z^{d-4} + x^d)((x - z)^2 + (y - z)^2) = 0
\]  \tag{2.2.5}

respectively. These curves have an \( \mathcal{E}_6 \)-node (respectively \( \mathcal{E}_7 \)-node) at \([0 : 0 : 1]\) and an \( \mathcal{A}_1 \)-node at another point.\(^2\) The base locus of this family is \([0 : 0 : 1]\). This argument works provided \( d \geq 6 = C_{\mathcal{A}_1} \mathcal{E}_6 = C_{\mathcal{A}_1} \mathcal{E}_7 \). This completes the proof of Lemma 2.2.3. \( \square \)

The rest of the proof of Lemma 2.7 in [2] is same as the proof of Lemma 2.6 in [1]. \( \square \)

### 2.3 Transversality of bundle sections

In this section we give proofs of Propositions 5.3 to 5.9 in [2]. Let us first recall the definition of vertical derivative.

**Definition 2.3.1.** Let \( \pi : V \to M \) be a holomorphic vector bundle of rank \( k \) and \( s : M \to V \) be a holomorphic section. Suppose \( h : V|_U \to U \times \mathbb{C}^k \) is a holomorphic trivialization of \( V \) and \( \pi_1, \pi_2 : U \times \mathbb{C}^k \to U, \mathbb{C}^k \) the projection maps. Let

\[
\hat{s} := \pi_2 \circ h \circ s. \tag{2.3.1}
\]

For \( q \in U \), we define the vertical derivative of \( s \) to be the \( \mathbb{C} \)-linear map

\[
\nabla s|_q : T_q M \to V_q, \quad \nabla s|_q := (\pi_2 \circ h)|_{V_q}^{-1} \circ d\hat{s}|_q,
\]

where \( V_q = \pi^{-1}(q) \), the fibre at \( q \). In particular, if \( v \in T_q M \) is given by a holomorphic map \( \gamma : B_\varepsilon(0) \to M \) such that \( \gamma(0) = q \) and \( \frac{\partial \gamma}{\partial z}|_{z=0} = v \), then

\[
\nabla s|_q(v) := (\pi_2 \circ h)|_{V_q}^{-1} \circ \frac{\partial \hat{s}(\gamma(z))}{\partial z}|_{z=0}
\]

were \( B_\varepsilon \) is an open \( \varepsilon \)-ball in \( \mathbb{C} \) around the origin.\(^3\) Finally, if \( v, w \in T_q M \) are tangent vectors such that there exists a family of complex curves \( \gamma : B_\varepsilon \times B_\varepsilon \to M \) such that

\[
\gamma(0, 0) = q, \quad \frac{\partial \gamma(x, y)}{\partial x}|_{(0,0)} = v, \quad \frac{\partial \gamma(x, y)}{\partial y}|_{(0,0)} = w
\]

then

\[
\nabla^{i+j} s|_q(v, \ldots, v, w, \ldots, w) := (\pi_2 \circ h)|_{V_q}^{-1} \circ \left[ \frac{\partial^{i+j} \hat{s}(\gamma(x, y))}{\partial^i x \partial^j y} \right]|_{(0,0)}.
\tag{2.3.2}
\]

\(^2\)To see why the last curve has an \( \mathcal{E}_7 \)-node at \([0 : 0 : 1]\), apply Lemma 4.8 in [1].

\(^3\)Not every tangent vector is given by a holomorphic map; however combined with the fact that \( \nabla s|_p \) is \( \mathbb{C} \)-linear, this definition determines \( \nabla s|_p \) completely.
Remark 2.3.2. In general the quantity in (2.3.2) is not well defined, i.e., it depends on the trivialization and the curve $\gamma$. In [1], Lemma ?? we explain on what subspace this quantity is well defined.

Remark 2.3.3. The section $s : M \rightarrow V$ is transverse to the zero set if and only if the induced map

$$\tilde{s} := \hat{s} \circ \varphi_{U}^{-1} : \mathbb{C}^{m} \rightarrow \mathbb{C}^{k}$$

(2.3.3)

is transverse to the zero set in the usual calculus sense, where $\varphi_{U} : \mathcal{U} \rightarrow \mathbb{C}^{m}$ is a coordinate chart and $\hat{s}$ is as defined in (2.3.1).

We will now show that the relevant bundle sections defined in [2] are transverse to the zero set. Let $\mathcal{F} \cong \mathbb{C}^{d+1}$ denote the space of homogeneous polynomials of degree $d$ and $\mathcal{F}^{*}$ the subspace of non-zero polynomials. This can also be thought of as the space of polynomials in two variables of degree at most $d$. If $V \rightarrow M$ is any vector bundle then a section

$$\psi : D \times M \rightarrow \pi_{D}^{*} \gamma_{D}^{*} \otimes \pi_{M}^{*} V$$

induces a section

$$\hat{\psi} : \mathcal{F}^{*} \times M \rightarrow \pi_{M}^{*} V \quad \text{given by} \quad \hat{\psi}(f, x) := \{\psi(\tilde{f}, x)\}(f).$$

(2.3.4)

We note that $\psi$ is transverse to zero at $(\tilde{f}, x)$ if and only if $\hat{\psi}$ is transverse to zero at $(f, x)$.

Proof of Proposition 5.3, [2]: To show transversality, we will use definition 2.3.1, remark 2.3.3 and (2.3.4). Using a similar argument as employed in [1], subsection ??, showing that the section $\pi_{2}^{*} \bar{\psi}_{A_{0}}$ is transverse to the zero set is equivalent to showing that the induced map

$$\pi_{1}^{*} \bar{\psi}_{A_{0}} \oplus \pi_{1}^{*} \bar{\psi}_{A_{1}} \oplus \pi_{2}^{*} \bar{\psi}_{A_{0}} : \mathcal{F}^{*} \times (\mathbb{C}^{2} \times \mathbb{C}^{2} - \Delta) \rightarrow \mathbb{C}^{4} \quad \text{given by}$$

$$\pi_{1}^{*} \bar{\psi}_{A_{0}} \oplus \pi_{1}^{*} \bar{\psi}_{A_{1}} \oplus \pi_{2}^{*} \bar{\psi}_{A_{0}}(f, x_{1}, y_{1}, x_{2}, y_{2}) := f(x_{1}, y_{1}), \; f_{x_{1}}(x_{1}, y_{1}), \; f_{y_{1}}(x_{1}, y_{1}), \; f(x_{2}, y_{2})$$

is transverse to the zero set at $(f, x_{1}, y_{1}, 0, 0)$ (in the usual calculus sense). We have seen (cf. Corollary ?? in [1]) that $\pi_{1}^{*} \bar{\psi}_{A_{0}} \oplus \pi_{1}^{*} \bar{\psi}_{A_{1}}$ is transverse to the zero set. Hence, $(\pi_{1}^{*} \bar{\psi}_{A_{0}} \oplus \pi_{1}^{*} \bar{\psi}_{A_{1}})^{-1}(0)$ is a smooth manifold. Let $(f, x_{1}, y_{1}, 0, 0)$ belong to the zero set of $\pi_{1}^{*} \bar{\psi}_{A_{0}} \oplus \pi_{1}^{*} \bar{\psi}_{A_{1}} \oplus \pi_{2}^{*} \bar{\psi}_{A_{0}}$. Hence

$$f(x_{1}, y_{1}) = 0, \; \nabla f|_{(x_{1}, y_{1})} = 0, \; f(0, 0) = 0.$$

Let $(\gamma(t), \gamma_{1}(t), \gamma_{2}(t))$ be a curve passing through $(f, x_{1}, y_{1}, 0, 0)$ at $t = 0$. This means

$$\gamma(t)(\gamma_{1}(t)) = 0, \; \nabla \gamma(t)|_{\gamma_{1}(t)} = 0.$$  

(2.3.5)

Restricted to $(\pi_{1}^{*} \bar{\psi}_{A_{0}} \oplus \pi_{1}^{*} \bar{\psi}_{A_{1}})^{-1}(0)$, the differential of the map $\pi_{2}^{*} \bar{\psi}_{A_{0}}$

$$D(\pi_{2}^{*} \bar{\psi}_{A_{0}}) : T_{(f, x_{1}, y_{1}, 0, 0)}((\pi_{1}^{*} \bar{\psi}_{A_{0}} \oplus \pi_{1}^{*} \bar{\psi}_{A_{1}})^{-1}(0)) \rightarrow T_{0}\mathbb{C} \cong \mathbb{C}$$

is given by

$$\langle \gamma'(0), \gamma_{1}'(0), \gamma_{2}'(0) \rangle \xrightarrow{D(\pi_{2}^{*} \bar{\psi}_{A_{0}})} \frac{d}{dt}|_{t=0} \gamma(t)(\gamma_{2}(t))$$

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Using Taylor expansion of $\gamma(t)$ and $\gamma_2(t)$ we see that

$$
\left. \frac{d}{dt} \right|_{t=0} \gamma(t)(\gamma_2(t)) = \nabla f|_{(0,0)} \cdot \gamma_2'(0) + \gamma'(0)(\gamma_2(0)).
$$

Therefore, for $D(\pi^*_2\psi_{A_0})$ to be surjective, it suffices to set $\gamma_1(t) := (x_1, y_1)$, $\gamma_2(t) := (0, 0)$ and $\gamma(t) := \gamma(0) + t\eta$ such that

$$
\eta(x_1, y_1) = 0, \quad \nabla \eta|_{(x_1, y_1)} = 0, \quad \eta(0, 0) \neq 0.
$$

This is to ensure that $(\gamma(t), \gamma_1(t), \gamma_2(t))$, as defined above, satisfies (2.3.5). A polynomial $\eta$ satisfying (2.3.6) exists if $d \geq 2$; we may set

$$
\eta(x, y) := \begin{cases} (x - x_1)^2 & \text{if } x_1 \neq 0, \\ (y - y_1)^2 & \text{if } x_1 = 0. \end{cases}
$$

This shows that $\pi^*_2\psi_{A_0}$ is transverse to the zero set and $\pi^*_2\psi_{A_0}^{-1}(0)$ is a smooth manifold.

Next, to show that $\pi^*_2\psi_{A_1}$ is transverse to the zero set, we need to show that the induced map

$$
\pi^*_1\psi_{A_0} \oplus \pi^*_1\psi_{A_1} \oplus \pi^*_2\psi_{A_0} \oplus \pi^*_2\psi_{A_1} : \mathcal{F}^* \times (\mathbb{C}^2 \times \mathbb{C}^2 - \Delta) \longrightarrow \mathbb{C}^6
$$

is transverse to the zero set at $(f, x_1, y_1, x_2, y_2)$ (in the usual calculus sense). As before, let $(\gamma(t), \gamma_1(t), \gamma_2(t))$ be a curve in $(\pi^*_1\psi_{A_0} \oplus \pi^*_1\psi_{A_1})^{-1}(0)$ at the point $(f, x_1, y_1, 0, 0)$. Restricted to $(\pi^*_1\psi_{A_0} \oplus \pi^*_1\psi_{A_1})^{-1}(0)$, the differential of the map $\pi^*_2\psi_{A_0} \oplus \pi^*_2\psi_{A_1}$ is given by

$$
D(\pi^*_2\psi_{A_0} \oplus \pi^*_2\psi_{A_1}) : T_{(f, x_1, y_1, 0, 0)}((\pi^*_1\psi_{A_0} \oplus \pi^*_1\psi_{A_1})^{-1}(0)) \longrightarrow T_0\mathbb{C}^3 \cong \mathbb{C}^3
$$

given by

$$
(\gamma'(0), \gamma_1'(0), \gamma_2'(0)) \xrightarrow{D(\pi^*_2\psi_{A_0} \oplus \pi^*_2\psi_{A_1})} \left( \nabla f|_{(0,0)} \cdot \gamma_2'(0) + \gamma'(0)(\gamma_2(0)), \quad \text{Hess } f|_{(0,0)} \cdot \gamma_2'(0) + \nabla \gamma'(0)|_{(0,0)} \right).
$$

Therefore, for $D(\pi^*_2\psi_{A_0} \oplus \pi^*_2\psi_{A_1})$ to be surjective, for any given vector $(v_0, v_{10}, v_{01}) \in T_0\mathbb{C}^3 \cong \mathbb{C}^3$, it suffices to set $\gamma_1(t) := (x_1, y_1)$, $\gamma_2(t) := (0, 0)$ and $\gamma(t) := \gamma(0) + t\eta$ such that $\eta$ satisfies

$$
\eta(x_1, y_1) = 0, \quad \nabla \eta|_{(x_1, y_1)} = 0, \quad \eta(0, 0) = v_0, \quad \nabla \eta|_{(0,0)} = (v_{10}, v_{01}).
$$

(2.3.7)

Such an $\eta$ always exists if $d \geq 3$. To see why this is so, first let us write $\eta$ as

$$
\eta(x, y) := \eta_1(x, y)\xi(x, y), \quad \xi(x, y) := \xi_{00} + \xi_{10}x + \xi_{01}y + \frac{\xi_{20}}{2}x^2 + \xi_{11}xy + \frac{\xi_{21}}{2}y^2 + \ldots
$$

(2.3.8)

where $\eta_1(x, y)$ is a polynomial satisfying (2.3.6) and $\xi(x, y)$ is a polynomial yet to be determined. If we can show $\xi$ exists, then we have shown $\eta$ exists. First note that since $\eta_1$ satisfies (2.3.6), $\eta$ satisfies the first two conditions of (2.3.7). The last two equations of (2.3.7) are equivalent to

$$
\begin{pmatrix}
\eta_1(0, 0) & 0 & 0 & \cdots \\
\ast & \eta_1(0, 0) & 0 & \cdots \\
\ast & 0 & \eta_1(0, 0) & \cdots
\end{pmatrix} \cdot \begin{pmatrix}
\xi_{00} & \xi_{10} & \xi_{01} & \cdots \\
\xi_{00} & \xi_{10} & \xi_{01} & \cdots \\
\xi_{00} & \xi_{10} & \xi_{01} & \cdots \\
\end{pmatrix}^T = \begin{pmatrix}
\mathbf{v}_0 \\
\mathbf{v}_{10} \\
\mathbf{v}_{01}
\end{pmatrix}.
$$

(2.3.9)
Since \( \eta_1 \) satisfies (2.3.6), \( \eta_1(0,0) \neq 0 \). Hence there exists a unique \((\xi_{00}, \xi_{10}, \xi_{01})\) solving (2.3.9). As a result \( \xi \) exists, which in turn implies \( \eta \) exists. \( \square \)

**Proof of Proposition 5.4, [2]:** Following a similar argument as before, showing that \( \pi_2^* \Psi_{P,A_2} \) is transverse to the zero set is equivalent to showing that the induced map

\[
\pi_1^* \psi_{A_0} \oplus \pi_1^* \psi_{A_1} \oplus \pi_2^* \psi_{A_0} \oplus \pi_2^* \psi_{A_1} \oplus \pi_2^* \psi_{P,A_2} : \mathcal{F}^* \times (\mathbb{C}^2 \times \mathbb{C}^2 - \Delta) \times \mathbb{C} \rightarrow \mathbb{C}^7
\]

given by

\[
(f, x_1, y_1, x_2, y_2, \theta) \rightarrow \left( f(x_1, y_1), f(x_1(x_1, y_1), f(x_2, y_2), f(x_2(x_2, y_2), f(y_2(x_2, y_2), f_{xx}(x_2, y_2) + \theta f_{xy}(x_2, y_2), f_{xy}(x_2, y_2) + \theta f_{yy}(x_2, y_2))\right)
\]

is transverse to the zero set at \((f, x_1, y_1, 0, 0, \theta)\) (in the usual calculus sense). Following a similar argument as before, showing that \( \eta(x, y) \) satisfies (2.3.8) as in (2.3.8). As before, \( \eta(x, y) \) satisfies the first two conditions of (2.3.10). The last four equations of (2.3.10) are equivalent to

\[
\begin{pmatrix}
\eta_1(0,0) & 0 & 0 & 0 & 0 & \cdots \\
* & \eta_1(0,0) & 0 & 0 & 0 & \cdots \\
* & 0 & \eta_1(0,0) & 0 & 0 & \cdots \\
* & * & * & \eta_1(0,0) & \theta & \cdots \\
* & * & * & 0 & \eta_1(0,0) & \cdots 
\end{pmatrix} \cdot (\xi_{00}, \xi_{10}, \xi_{01}, \xi_{20}, \xi_{11}, \cdots \xi_{0d})^T = \begin{pmatrix} \mathbf{v}_0 \\ \mathbf{v}_{10} \\ \mathbf{v}_{01} \\ \mathbf{v}_{20} \\ \mathbf{v}_{11} \end{pmatrix}.
\]

Since \( \eta_1(0,0) \neq 0 \), there exists a unique \((\xi_{00}, \xi_{10}, \xi_{01}, \xi_{20}, \xi_{11})\) solving the above equation. As a result \( \xi \) exists, which in turn implies \( \eta \) exists. \( \square \)

**Proof of Proposition 5.5, [2]:** Following a similar argument as before, showing that \( \pi_2^* \Psi_{P,A_3} \) is transverse to the zero set is equivalent to showing that the induced map

\[
\pi_1^* \psi_{A_0} \oplus \pi_1^* \psi_{A_1} \oplus \pi_2^* \psi_{A_0} \oplus \pi_2^* \psi_{A_1} \oplus \pi_2^* \psi_{P,A_2} \oplus \pi_2^* \psi_{P,A_3} : \mathcal{F}^* \times (\mathbb{C}^2 \times \mathbb{C}^2 - \Delta) \times \mathbb{C} \rightarrow \mathbb{C}^8
\]

given by

\[
(f, x_1, y_1, x_2, y_2, \theta) \rightarrow \left( f(x_1, y_1), f(x_1(x_1, y_1), f(x_2, y_2), f(x_2(x_2, y_2), f(y_2(x_2, y_2), f_{xx}(x_2, y_2) + \theta f_{xy}(x_2, y_2), f_{xy}(x_2, y_2) + \theta f_{yy}(x_2, y_2), f_{x\hat{x}\hat{x}}(x_2, y_2))\right)
\]

is transverse to the zero set at \((f, x_1, y_1, 0, 0, \theta)\), where \( \hat{x} := x + \theta y \). Following a similar argument as before, this is equivalent to showing that given a \((\mathbf{v}_0, \mathbf{v}_{10}, \mathbf{v}_{01}, \mathbf{v}_{20}, \mathbf{v}_{11}, \mathbf{v}_{30})\) \in \mathbb{C}^6, there exists a polynomial \( \eta(x, y) \) such that

\[
\eta(x_1, y_1) = 0, \quad \nabla \eta|_{(x_1, y_1)} = (0, 0), \quad \eta(0, 0) = \mathbf{v}_0, \quad \nabla \eta|_{(0,0)} = (\mathbf{v}_{10}, \mathbf{v}_{01}),
\]

\[
\eta_{xx}(0,0) + \theta \eta_{xy}(0,0) = \mathbf{v}_{20}, \quad \eta_{xy}(0,0) + \theta \eta_{yy}(0,0) = \mathbf{v}_{11}, \quad \eta_{x\hat{x}\hat{x}} = \mathbf{v}_{30}.
\]

Such an \( \eta \) always exists if \( d \geq 5 \). To see why, let us write \( \eta \) as in (2.3.8). As before, \( \eta(x, y) \) satisfies

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we define a series of numbers $A_k(x, y)$. The algorithm to obtain these numbers $A_k(x, y)$ is given in [1], Lemma 4.4. They are also explicitly written down in [2], [1] till $k = 7$. Next, given a polynomials $f(x, y)$ and $\eta(x, y)$, define

$$
\hat{f}(\hat{x}, y) := f(\hat{x} - \theta y, y) \quad \text{and} \quad J_{A_k} := \lim_{t \to 0} \frac{\hat{f}_{02} + t\eta_{02}}{t}k^{-3}A_k^{f + t\eta} - \hat{f}_{02}k^{-3}A_k^{f}.
$$

Since $\eta(0, 0) \neq 0$, there exists a unique $(\xi_{00}, \xi_{10}, \xi_{01}, \xi_{20}, \xi_{21}, \xi_{30})$ solving the above equation. As a result $\xi$ exists, which in turn implies $\eta$ exists.

By following a similar argument, to show that $\pi_2^*\Psi_{P_D}$ is transverse to the zero set, we need to show that given a $(v_0, v_{10}, v_{01}, v_{20}, v_{11}, v_{30}, v_{02}) \in \mathbb{C}^7$, there exists a polynomial $\eta(x, y)$ such that

$$
\eta(x_1, y_1) = 0, \quad \nabla \eta(x_1, y_1) = (0, 0), \quad \eta(0, 0) = v_0, \quad \nabla \eta(0, 0) = (v_{10}, v_{01}),
$$

$$
\eta_{xx}(0, 0) + \theta \eta_{xy}(0, 0) = v_{20}, \quad \eta_{xy}(0, 0) + \theta \eta_{yy}(0, 0) = v_{11}, \quad \eta_{xx} = v_{30}, \quad \eta_{yy}(0, 0) = v_{02},
$$

where $\hat{x} := x + \theta y$. Such an $\eta$ always exists if $d \geq 5$. To see why, again, write $\eta$ as in (2.3.8). As before, $\eta(x, y)$ satisfies the first two conditions of (2.3.12). The last six equations of (2.3.12) always have a solution; it is easy to see that there exists a unique $(a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{30}, a_{02})$ that solves the last six equations of (2.3.12). As a result $\xi$ exists, which in turn implies $\eta$ exists.

Finally to show that $\pi_2^*\Psi_{P_D}$ is transverse to the zero set, we need to show that given a $(v_0, v_{10}, v_{01}, v_{20}, v_{11}, v_{30}, v_{02}, v_{21}) \in \mathbb{C}^8$, there exists a polynomial $\eta(x, y)$ such that

$$
\eta(x_1, y_1) = 0, \quad \nabla \eta(x_1, y_1) = (0, 0), \quad \eta(0, 0) = v_0, \quad \nabla \eta(0, 0) = (v_{10}, v_{01}),
$$

$$
\eta_{xx}(0, 0) + \theta \eta_{xy}(0, 0) = v_{20}, \quad \eta_{xy}(0, 0) + \theta \eta_{yy}(0, 0) = v_{11}, \quad \eta_{xx} = v_{30}, \quad \eta_{yy}(0, 0) = v_{02}, \quad \eta_{xx}(0, 0) = v_{21}
$$

where $\hat{x} := x + \theta y$. Such an $\eta$ always exists if $d \geq 5$. To see why, again, write $\eta$ as in (2.3.8). As before, $\eta(x, y)$ satisfies the first two conditions of (2.13). The last seven equations of (2.12) always have a solution; it is easy to see that there exists a unique $(\xi_{00}, \xi_{10}, \xi_{01}, \xi_{20}, \xi_{21}, \xi_{30}, \xi_{02})$ that solves the last six equations of (2.12). As a result $\xi$ exists, which in turn implies $\eta$ exists.

**Proof of Proposition 5.6, [2]**: Recall that given a polynomial $f(x, y)$ and a point $(x_0, y_0) \in \mathbb{C}^2$, we define a series of numbers $A_k^f(x_0, y_0)$, which are functions of

$$
f_{ij}(x_0, y_0) := \frac{\partial^{i+j} f(x, y)}{\partial^i x \partial^j y} \bigg|_{(x, y) = (x_0, y_0)}.
$$

The algorithm to obtain these numbers $A_k^f$ is given in [1], Lemma 4.4. They are also explicitly written down in [2], [2] till $k = 7$. Next, given a polynomials $f(x, y)$ and $\eta(x, y)$, define

$$
\hat{f}(\hat{x}, y) := f(\hat{x} - \theta y, y) \quad \text{and} \quad J_{A_k} := \lim_{t \to 0} \frac{\hat{f}_{02} + t\eta_{02}}{t}k^{-3}A_k^{f + t\eta} - \hat{f}_{02}k^{-3}A_k^{f}.
$$

---

4 There the point $(x_0, y_0)$ was taken to be $(0, 0)$; to obtain $A_k^f(x_0, y_0)$ replace $f_{ij}$ by $f_{ij}(x_0, y_0)$ in $A_k^f$. 

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Here $\theta$ is any complex number. To show that our sections are transverse to the zero set, we need to show that the induced map

$$\pi_1^\top \Psi_{A_0} + \pi_1^\top \Psi_{A_1} + \pi_2^\top \Psi_{A_0} + \pi_2^\top \Psi_{A_1} + \pi_3^\top \Psi_{P_{A_0}} + \pi_3^\top \Psi_{P_{A_1}} + \ldots + \pi_n^\top \Psi_{P_{A_i}} : F^* \times (C^2 \times C^2 - \Delta) \times C \rightarrow \mathbb{C}^{i+5},$$

is transverse to the zero set at $(f, x_1, y_1, 0, 0, \theta)$, provided $f_{02}(0, 0) \neq 0$ (this corresponds to the condition $\pi_5^\top \Psi_{P_{D_k}}(f, \bar{p}, l_p) \neq 0$). Here $\hat{x} := x + \theta y$. Following a similar argument as before, we need to show that given a $(\nu_0, \nu_{10}, \nu_{01}, \nu_{20}, \nu_{11}, \nu_{30}, \ldots, \nu_{i0}) \in \mathbb{C}^{i+5}$, there exists a polynomial $\eta(x, y)$ such that

$$\eta(x_1, y_1) = 0, \quad \nabla \eta|_{(x_1, y_1)} = (0, 0), \quad \eta(0, 0) = \nu_0, \quad \nabla \eta|_{(0,0)} = (\nu_{10}, \nu_{01}),$$

$$\eta_{xx}(0, 0) + \theta \eta_{xy}(0, 0) = \nu_{20}, \quad \eta_{xy}(0, 0) + \theta \eta_{yy}(0, 0) = \nu_{11},$$

$$\eta_{xx}(0, 0) = \nu_{30}, \quad \mathcal{J}_{A_k} = 0, \quad \forall \quad 4 \leq k \leq i. \quad (2.3.15)$$

Such an $\eta$ always exists if $d \geq i+2$. To see why, again, write $\eta$ as in (2.3.8). As before, $\eta(x, y)$ satisfies the first two conditions of (2.3.15). The last $i+2$ equations of (2.3.15) always have a solution; since $f_{02}(0, 0) \neq 0$ it is easy to see that there exists a unique $(\xi_{00}, \xi_{01}, \xi_{20}, \xi_{11}, \xi_{30}, \ldots, \xi_{i0})$ that solves the last $i+2$ equations of (2.3.15). As a result $\xi$ exists, which in turn implies $\eta$ exists.

**Proof of Proposition 5.7, [2]:** To show that $\pi_5^\top \Psi_{P_{D_k}}$ is transverse to the zero set we need to show that given a $(\nu_0, \nu_{10}, \nu_{01}, \nu_{20}, \nu_{11}, \nu_{30}, \nu_{02}, \nu_{21}, \nu_{40}) \in \mathbb{C}^9$, there exists a polynomial $\eta(x, y)$ such that

$$\eta(x_1, y_1) = 0, \quad \nabla \eta|_{(x_1, y_1)} = (0, 0), \quad \eta(0, 0) = \nu_0, \quad \nabla \eta|_{(0,0)} = (\nu_{10}, \nu_{01}),$$

$$\eta_{xx}(0, 0) + \theta \eta_{xy}(0, 0) = \nu_{20}, \quad \eta_{xy}(0, 0) + \theta \eta_{yy}(0, 0) = \nu_{11},$$

$$\eta_{xx}(0, 0) = \nu_{30}, \quad \eta_{yy}(0, 0) = \nu_{02}, \quad \eta_{xx}(0, 0) = \nu_{21}, \quad \eta_{xx}(0, 0) = \nu_{40} \quad (2.3.16)$$

where $\hat{x} := x + \theta y$. Such an $\eta$ always exists if $d \geq 6$. To see why, write $\eta$ as in (2.3.8). As before, $\eta(x, y)$ satisfies the first two conditions of (2.3.16). The last eight equations of (2.3.16) always have a solution; it is easy to see that there exists a unique $(\xi_{00}, \xi_{01}, \xi_{0,0}, \xi_{20}, \xi_{11}, \xi_{30}, \xi_{02}, \xi_{21}, \xi_{40})$ that solves the last eight equations of (2.3.16). As a result $\xi$ exists, which in turn implies $\eta$ exists.

To show that $\pi_5^\top \Psi_{P_{D_k}}$ is transverse to the zero set for $i \geq 7$, we use a similar argument as in the proof of Proposition 5.6, [2]. Recall that given a polynomial $f(x, y)$ and a point $(x_0, y_0) \in \mathbb{C}^2$, we define a series of numbers $D_k^f(x_0, y_0)$. The algorithm to obtain these numbers $D_k^f$ is given [1], Lemma 4.6. They are also explicitly written down in (??) [1] till $k = 8$. Given polynomials $f(x, y)$ and $\eta(x, y)$, let us define

$$\hat{f}(\hat{x}, y) := f(\hat{x} - \theta y, y) \quad \text{and} \quad \mathcal{J}_{D_k} := \lim_{t \to 0} \frac{(\hat{f}_{02} + t \eta_{02})^{k-3} \hat{D}_k^{f+t\eta} - \hat{f}_{02}^{k-3} \hat{D}_k^f}{t} \quad (2.3.17)$$

where $\theta$ is any complex number. Following a similar argument as in the proof of Proposition 5.6, [2], to show transversality, we need to show that given a

$$\nu_0, \nu_{10}, \nu_{01}, \nu_{20}, \nu_{11}, \nu_{30}, \nu_{02}, \nu_{21}, \nu_{40}, \ldots, \nu_{i0} \in \mathbb{C}^{i+5},$$

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a polynomial $\eta(x, y)$ exists that satisfies

\[
\begin{align*}
\eta(x_1, y_1) &= 0, \quad \nabla \eta|_{(x_1, y_1)} = (0, 0), \quad \eta(0, 0) = v_0, \quad \nabla \eta|_{(0, 0)} = (v_{10}, v_{01}), \\
\eta_{xx}(0, 0) + \theta \eta_{xy}(0, 0) &= v_{20}, \quad \eta_{xy}(0, 0) + \theta \eta_{yy}(0, 0) = v_{11}, \\
\eta_{x^2x^2}(0, 0) &= v_{30}, \quad \eta_{yy}(0, 0) = v_{02}, \quad \eta_{x^2y}(0, 0) = v_{21}, \quad \eta_{x^2x^2y} = v_{40}, \quad \mathcal{J}_{D_k} = 0 \quad \forall \quad 7 \leq k \leq i
\end{align*}
\]

where $\hat{x} := x + \theta y$. Such an $\eta$ always exists if $d \geq i + 2$. To see why, write $\eta$ as in (2.3.8). As before, $\eta(x, y)$ satisfies the first two conditions of (2.3.18). Since $\pi^* \Psi_{PC \infty}(f, \tilde{p}, l_p) \neq 0$, we conclude $\pi_{12} \neq 0$. Hence, the last $i + 4$ equations of (2.3.18) always have a solution; it is easy to see that there exists a unique $(\xi_{00}, \xi_{10}, \xi_{01}, \xi_{20}, \xi_{11}, \xi_{30}, \xi_{02}, \xi_{21}, \xi_{31}, \xi_{03}, \xi_{22}, \xi_{32}, \xi_{04}, \xi_{23}, \xi_{33}, \xi_{05}, \xi_{24}, \xi_{34}, \xi_{06}, \xi_{25}, \xi_{35}, \xi_{07}, \xi_{26}, \xi_{36}, \xi_{08}, \xi_{27}, \xi_{37}, \xi_{09}, \xi_{28}, \xi_{38}, \xi_{010}, \xi_{29}, \xi_{39}, \xi_{011}, \xi_{210}, \xi_{311}, \xi_{012}, \xi_{211}, \xi_{312})$ that solves the last $i + 4$ equations of (2.3.18). As a result $\xi$ exists, which in turn implies $\eta$ exists.

Proof of Proposition 5.8, [2]: To show that $\pi^* \Psi_{PC \infty}$ is transverse to the zero set we need to show that given a $(v_0, v_{10}, v_{01}, v_{20}, v_{11}, v_{30}, v_{02}, v_{21}, v_{12}) \in \mathbb{C}^9$, there exists a polynomial $\eta(x, y)$ such that

\[
\begin{align*}
\eta(x_1, y_1) &= 0, \quad \nabla \eta|_{(x_1, y_1)} = (0, 0), \quad \eta(0, 0) = v_0, \quad \nabla \eta|_{(0, 0)} = (v_{10}, v_{01}), \\
\eta_{xx}(0, 0) + \theta \eta_{xy}(0, 0) &= v_{20}, \quad \eta_{xy}(0, 0) + \theta \eta_{yy}(0, 0) = v_{11}, \\
\eta_{x^2x^2}(0, 0) &= v_{30}, \quad \eta_{yy}(0, 0) = v_{02}, \quad \eta_{x^2y}(0, 0) = v_{21}, \quad \eta_{x^2x^2y} = v_{12}
\end{align*}
\]

where $\hat{x} := x + \theta y$. Such an $\eta$ always exists if $d \geq 5$. To see why, write $\eta$ as in (2.3.8). As before, $\eta(x, y)$ satisfies the first two conditions of (2.3.19). The last eight equations of (2.3.19) always have a solution; it is easy to see that there exists a unique $(\xi_{00}, \xi_{10}, \xi_{01}, \xi_{20}, \xi_{11}, \xi_{30}, \xi_{02}, \xi_{21}, \xi_{31})$ that solves the last eight equations of (2.3.19). As a result $\xi$ exists, which in turn implies $\eta$ exists.

Proof of Proposition 5.9, [2]: To show that $\pi^* \Psi_{PA_3}$ is transverse to the zero set we need to show that given a $(v_0, v_{10}, v_{01}, v_{20}, v_{11}, v_{02}, v_{30}) \in \mathbb{C}^7$, and a complex number $\theta$, there exists a polynomial $\eta(x, y)$ such that

\[
\begin{align*}
\eta(x_1, y_1) &= 0, \quad \nabla \eta|_{(x_1, y_1)} = (0, 0), \quad \eta(0, 0) = v_0, \quad \nabla \eta|_{(0, 0)} = (v_{10}, v_{01}), \\
\eta_{xx}(0, 0) &= v_{20}, \quad \eta_{xy}(0, 0) = v_{11}, \quad \eta_{yy}(0, 0) = v_{02}, \quad \eta_{x^2x^2y} = v_{30}, \quad \eta_{x^2x^2y} = v_{30}, \quad \eta_{x^2x^2y} = v_{30}, \quad \eta_{x^2x^2y} = v_{30}, \quad \eta_{x^2x^2y} = v_{30}, \quad \eta_{x^2x^2y} = v_{30}, \quad \eta_{x^2x^2y} = v_{30}, \quad \eta_{x^2x^2y} = v_{30}, \quad \eta_{x^2x^2y} = v_{30}, \quad \eta_{x^2x^2y} = v_{30}, \quad \eta_{x^2x^2y} = v_{30}, \quad \eta_{x^2x^2y} = v_{30}
\end{align*}
\]

where $\hat{x} := x + \theta y$. Such an $\eta$ always exists if $d \geq 5$. To see why, write $\eta$ as in (2.3.8). As before, $\eta(x, y)$ satisfies the first two conditions of (2.3.20). The last six equations of (2.3.19) always have a solution; it is easy to see that there exists a unique $(\xi_{00}, \xi_{10}, \xi_{01}, \xi_{20}, \xi_{11}, \xi_{02}, \xi_{30})$ that solves the last six equations of (2.3.20). As a result $\xi$ exists, which in turn implies $\eta$ exists.

2.4 Low degree checks

2.4.1 Verification of the number $\mathcal{N}(A_1 PD_d, n, m)$

In this subsection we will verify the number $\mathcal{N}(A_1 PD_d, n, m)$ for $d = 4$ for all values of $m$ and $n$. First of all note that a quartic will have a one node and a $D_4$-node, if and only if the quartic breaks into a nodal cubic and a line, such that the line passes through the node. Let us now consider the
To compute $A_{m,n,k}$

Note that dimension of $M$ through the node of the cubic. A little bit of thought will reveal that this number is $A_{\text{line that passes through that node and the third derivative along the three distinguished directions}}$.

how many different ways the quartic passing through nine points can split into a nodal cubic and a

$\text{can pass through 14 points}$. Hence, this quartic passes through 9 points. We wish to compute in

f cubics $N$.

We are now ready to compute the numbers $\ldots$ other two directions, i.e. the two directions along which the second derivative of the cubic vanishes.

It is easy to see that $\ldots$ appropriately as the zero set of a vector bundle, we can see that

$PD_M(Z_1) = e(\gamma_{D_1} \otimes \gamma_{p_2}^*) e(\gamma_{D_3} \otimes \gamma_{p_2}^* \otimes T^*\mathbb{P}^2) e(\gamma_{D_1} \otimes \gamma_{p_2}^* \otimes \gamma^*)$

$PD_M(Z_3) = e(\gamma_{D_1}^* \otimes \gamma_{p_2}^*) e(\gamma_{D_3} \otimes \gamma_{p_2}^* \otimes T^*\mathbb{P}^2) e(\gamma_{D_3} \otimes \gamma_{p_2}^* \otimes \gamma^*)$

where $M := D_1 \times D_3 \times \mathbb{PT}^2$.

It is easy to see that $Z_1$ is the space of nodal cubics $f_3$ and a line $f_1$, such that third derivative of the resulting quartic $f_3f_1$ vanishes along the direction of the line. Similarly, $Z_3$ is the space of nodal cubics $f_3$ and a line $f_1$, such that third derivative of the resulting quartic $f_3f_1$ vanishes along the other two directions, i.e. the two directions along which the second derivative of the cubic vanishes.

We are now ready to compute the numbers $N(A_1PD_4, n, m)$ for $d = 4$.

Let us denote the first Chern classes of $\gamma_{D_1}^*$, $\gamma_{D_3}^*$, $\gamma_{p_2}^*$ and $\gamma^*$ as $y_1$, $y_3$, $a$ and $\lambda$ respectively. Note that

$\lambda^2 = -3a\lambda - 3a^2$, \quad $\lambda^3 = 6a^2\lambda$, \quad $\lambda^n = 0 \quad \forall \ n \geq 4$.

Note that dimension of $M$ is 14. Let

$A_{m,n,k} := (y_3^{9-(n+m+k)} y_1^a a^\lambda (PD_M Z_1 + PD_M Z_3), [M])$.

To compute $A_{m,n,k}$, let

$B_{m,n,k} := \text{Coefficient of } y_1^2 y_3^a a^2 \lambda \text{ in } y_3^{9-(n+m+k)} y_1^a a^\lambda (PD_M Z_1 + PD_M Z_3)$,

$C_{m,n,k} := \text{Coefficient of } y_1^2 y_3^a a^2 \lambda \text{ in } y_3^{9-(n+m+k)} y_1^a a^\lambda (PD_M Z_1 + PD_M Z_3)$,

$D_{m,n,k} := \text{Coefficient of } y_1^2 y_3^a a^2 \lambda \text{ in } y_3^{9-(n+m+k)} y_1^a a^\lambda (PD_M Z_1 + PD_M Z_3)$.

It is easy to see that

$A_{m,n,k} = B_{m,n,k} - 3C_{m,n,k} + 6D_{m,n,k}$.

Suppose we wish to compute $N(A_1PD_4, 0, 0)$. This is a codimension 5 number. A generic quartic can pass through 14 points. Hence, this quartic passes through 9 points. We wish to compute in how many different ways the quartic passing through nine points can split into a nodal cubic and a line that passes through that node and the third derivative along the three distinguished directions is zero. First of all we can place the nodal cubic through the 9 points and place the remaining line through the node of the cubic. A little bit of thought will reveal that this number is $A_{0,0,0}$. This number is clearly zero, since there are no nodal cubics through 9 points. That is verified by observing
that $A_{0,0,0}$ is indeed zero. Next, we can choose any of the 8 points and place a nodal cubic through them and then we can put the line through the remaining point and the node of the cubic (keeping the conditions on the third derivatives satisfied). A little bit of thought will reveal that this number is $A_{0,0,1}$. Finally, we can place the line through 2 points and count how many nodal cubics are there through 7 points with the node on the line (again, keeping the conditions on the third derivatives satisfied). A little bit of thought will reveal that this number is $A_{0,0,2}$. Of course, we can keep continuing like this all the way, but all the subsequent numbers will be zero, since a line can not pass through three or more points. Hence, what we expect is that for $d = 4$,

$$N(A_1 \mathcal{PD}_4, 0, 0) = \binom{9}{9} A_{0,0,0} + \binom{9}{8} A_{0,0,1} + \binom{9}{7} A_{0,0,2} = \sum_{k=0}^{9} \left( \binom{9}{9-k} A_{0,0,k} \right) = 9^{72}.$$

This agrees with our formula for general $d$.

A similar argument now shows that for all $m$ and $n$, and $d = 4$:

$$N(A_1 \mathcal{PD}_4, n, m) = \sum_{k=0}^{9-(n+m)} \left( \binom{9-(n+m)}{9-(n+m+k)} A_{m,n,k} \right).$$

A straightforward calculation for each value of $m$ and $n$ now shows that we recover the answer we obtain in [2].

### 2.5 A few closure claims

**Lemma 2.5.1.** We have the following equality of sets

$$\overline{A}_1 = A_1 \cup \overline{A}_2 \quad (2.5.1)$$

if $d \geq 3$.

**Proof:** First of all recall that

$$A_1 = \{(\tilde{f}, \tilde{p}) \in \mathcal{D} \times \mathbb{P}^2 : f(p) = 0, \; \nabla f|_{\tilde{p}} = 0, \; \det \nabla^2 f|_{\tilde{p}} \neq 0 \},$$

$$\overline{A}_1 = \{(\tilde{f}, \tilde{p}) \in \mathcal{D} \times \mathbb{P}^2 : f(p) = 0, \; \nabla f|_{\tilde{p}} = 0 \}.$$

Hence, it suffices to show that

$$\overline{A}_2 = \{(\tilde{f}, \tilde{p}) \in \mathcal{D} \times \mathbb{P}^2 : f(p) = 0, \; \nabla f|_{\tilde{p}} = 0, \; \det \nabla^2 f|_{\tilde{p}} = 0 \}. \quad (2.5.2)$$

First, note that the lhs of (2.5.2) is a subset of its rhs. To show the converse, suppose $(\tilde{f}, \tilde{p})$ belongs to the rhs of (2.5.2). Let us write down $f$ in local coordinates, i.e.

$$f = \frac{f_{20}}{2} x^2 + f_{11}xy + \frac{f_{02}}{2} y^2 + \ldots \quad \text{such that} \quad f_{11}^2 - f_{20}f_{02} = 0.$$

We need to show that there exists a sequence $(\tilde{f}(t), \tilde{p}(t))$ in $A_2$ that converges to $(\tilde{f}, \tilde{p})$. We will just change the curve $\tilde{f}(t)$ and keep the point $\tilde{p}$ unchanged. First of all, if $(\tilde{f}, \tilde{p})$ belongs to $A_2$, then there is nothing to prove. Suppose $(\tilde{f}, \tilde{p})$ does not belong to $A_2$. Let us first assume that $\nabla^2 f|_{\tilde{p}}$ is
not identically zero. In that case \( f_{20} \) and \( f_{02} \) can not both be zero. Assume \( f_{02} \neq 0 \). Hence, by the implicit function theorem, there exists a change of coordinates

\[
y = \hat{y} + B(x)
\]

so that we can write \( f \) as

\[
f = \frac{f_{02}}{2} \hat{y}^2 + \frac{B_3^f}{6} x^3 + \ldots \quad \text{where} \quad B_3^f := f_{30} - \frac{3 f_{11} f_{21}}{f_{02}} + \frac{3 f_{12}^2}{f_{02}^2} - \frac{f_{11}^3 f_{03}}{f_{02}^3}.
\]

Now consider the following sequence

\[
f_{30}(t) = t + \frac{3 f_{11} f_{21}}{f_{02}} - \frac{3 f_{12}^2}{f_{02}^2} + \frac{f_{11}^2 f_{03}}{f_{02}^3},
\]

\[
f_{ij}(t) = f_{ij} \quad \text{if} \quad (i, j) \neq (3, 0).
\]

This sequence lies in \( A_2 \) \(^5\) converging to \((\hat{f}, \hat{p})\). Similar argument works if \( f_{02} = 0 \), but \( f_{20} \neq 0 \). Finally, suppose \( \nabla^2 f \big|_{\hat{p}} \) is identically zero. Then consider the sequence

\[
f_{02}(t) = t
\]

\[
f_{30}(t) = f_{30} + t
\]

\[
f_{ij}(t) = f_{ij} \quad \text{if} \quad (i, j) \neq (0, 2) \text{ or } (3, 0).
\]

This sequence lies in \( A_2 \) that converges to \((\hat{f}, \hat{p})\). This proves the claim.

\[\]

\[\]

**Lemma 2.5.2.** We have the following equality of sets

\[
\hat{D}^k_{\#} = \hat{D}_k.
\]

if \( k \geq 4 \) and \( d \geq 3 \).

**Proof:** Since \( \hat{D}^k_{\#} \subset \hat{D}_k \), the lhs of (2.5.3) is a subset of its rhs. To prove the converse, it suffices to show that

\[
\hat{D}^k_{\#} \supset \hat{D}_k.
\]

Let \((\hat{f}, \hat{l}) \in \hat{D}_k \). We need to show that there exists a sequence in \((\hat{f}(t), l_{p(t)}) \in \hat{D}^k_{\#} \) that converges to \((\hat{f}, \hat{l})\). If \((\hat{f}, \hat{l})\) already belongs to \( \hat{D}^k_{\#} \) then there is nothing to prove. So let us assume that \((\hat{f}, \hat{l})\) belongs to \( \hat{D}_k - \hat{D}^k_{\#} \). Let us write down \( f \) in local coordinates and let us assume the distinguished direction is \( \partial_x \), i.e. the third derivative along the \( x \) direction is zero. Hence, in local coordinates

\[
f = \frac{f_{30}}{6} x^3 + \frac{f_{21}}{2} x^2 y + \frac{f_{12}}{2} x y^2 + \frac{f_{03}}{6} y^3 + \ldots
\]

To construct the sequence \((\hat{f}(t), l_{p(t)}) \in \hat{D}^k_{\#} \), we will just change the direction and keep the curve and the point unchanged. We claim that if \( \eta \) is sufficiently small and non zero, then \((\partial_x + \eta \partial_y)^3 f \big|_{(0,0)}\) is non zero. To see why, note that

\[
(\partial_x + \eta \partial_y)^3 f \big|_{(0,0)} = f_{30} + \eta f_{21} + \eta^2 f_{12} + \eta^3 f_{03}.
\]

\[\]

\[\]

\[\]

---

\(^5\)If \( t \) is small then \( B_3^{\hat{f}(t)} \neq 0 \).
Since \( \tilde{f} \) has a \( D_k \)-node at \( \tilde{p} \), we conclude that \( f_{30}, f_{21}, f_{12} \) and \( f_{03} \) are not all zero. Hence if \( \eta \) is sufficiently small, the rhs of (2.5.5) is non zero; just choose \( \eta \) so that it is not a solution of \( f_{30} + \eta f_{21} + \eta^2 f_{12} + \eta^3 f_{03} = 0 \). That proves the claim. \( \square \)

**Lemma 2.5.3.** We have the following equality of sets

\[
\overline{D_k^p} = \overline{D_4}.
\] (2.5.6)

if \( d \geq 3 \).

**Proof:** By Corollary 1.6.18, it suffices to show that

\[
\overline{D_4} = \{(\tilde{f}, l_p) \in D \times \mathbb{P} \mathbb{T}^2 : f(p) = 0, \ \nabla f|_p = 0, \ \nabla^2 f|_p = 0\}. \tag{2.5.7}
\]

Suppose \((\tilde{f}, l_p)\) belongs to the rhs of (2.5.7), but not in \( \overline{D_4} \). We need to show that there exists a sequence \((\tilde{f}(t), l_{p(t)}) \in \overline{D_4}\) that converges to \((\tilde{f}, l_p)\). First we write down \( f \) in local coordinates

\[
f = \frac{f_{30}}{6} x^3 + \frac{f_{21}}{2} x^2 y + \frac{f_{12}}{2} xy^2 + \frac{f_{03}}{6} y^3 + \ldots
\]

Define

\[
\beta := f_{30}^2 f_{03}^2 - 6 f_{03} f_{12} f_{21} f_{30} + 4 f_{12}^3 f_{30} + 4 f_{03} f_{21}^3 - 3 f_{12}^2 f_{21}^2.
\]

Since \((\tilde{f}, l_p)\) belongs to the rhs of (2.5.7), but not in \( \overline{D_4} \), we conclude that \( \beta = 0 \) since \( \beta \) is the discriminant of the cubic term in the Taylor expansion of \( f \). Now we will construct the desired sequence \( f_{ij}(t) \).

First, let us assume \( f_{30} \neq 0 \). Define \( f_{03}(t) := f_{03} + t \) and \( f_{ij}(t) = f_{ij} \) if \((i, j) \neq (0, 3)\). It is easy to see that the corresponding value of \( \beta(t) \) (with \( f_{ij} \) replaced by \( f_{ij}(t) \) in \( \beta \)) is non zero if \( t \) is small but non zero. A similar sequence can be constructed if \( f_{30} = 0 \), but \( f_{03} \neq 0 \). Finally assume that \( f_{30} = f_{03} = 0 \). Let \( f_{12}(t) = f_{12} + t \), \( f_{21}(t) = f_{21} + t \) and \( f_{ij}(t) = f_{ij} \) if \((i, j) \neq (1, 2)\) or \((2, 1)\). It is easy to see that the corresponding value of \( \beta(t) \) is non zero if \( t \) is small but non zero. \( \square \)

**Lemma 2.5.4.** We have the following equality of sets

\[
\{(\tilde{f}, l_p) \in \overline{D_5} : \Psi_{P A_3}(\tilde{f}, l_p) = 0\} = \overline{PD_5} \cup \overline{PD_5^2}.
\] (2.5.8)

if \( d \geq 3 \).

**Proof:** Suppose \((\tilde{f}, l_p) \in \overline{D_5}\) and \( \Psi_{P A_3}(\tilde{f}, l_p) = 0 \). First we write down \( f \) in local coordinates

\[
f = \frac{f_{30}}{6} x^3 + \frac{f_{21}}{2} x^2 y + \frac{f_{12}}{2} xy^2 + \frac{f_{03}}{6} y^3 + \ldots
\]

Define

\[
\beta := f_{30}^2 f_{03}^2 - 6 f_{03} f_{12} f_{21} f_{30} + 4 f_{12}^3 f_{30} + 4 f_{03} f_{21}^3 - 3 f_{12}^2 f_{21}^2.
\]

First we note that if \((\tilde{f}, l_p) \in \overline{D_5} \) then \( \beta = 0 \). Next let us assume without loss of generality that the distinguished direction is \( \partial_x \in l_p \). Hence \( f_{30} = 0 \). Since \( \beta = 0 \), this implies that either \( f_{21} = 0 \) or \( 3 f_{12}^2 - 4 f_{21} f_{03} = 0 \). Next, we observe that \((\tilde{f}, l_p) \in \overline{PD_5} \) if and only if

\[
f_{30}, f_{21} = 0. \tag{2.5.9}
\]
This follows from Corollary 1.6.27. Finally, we observe that \((\tilde{f}, \tilde{p}) \in \overline{PD}_0^\gamma\) if and only if
\[
f_{30}, \quad 3f_{12}^2 - 4f_{21}f_{03} = 0.
\] (2.5.10)

This follows from Corollary 1.6.29. This proves our claim. □

**Lemma 2.5.5.** We have the following inclusion of sets
\[
\overline{PE}_6 \subset \overline{PD}_5^\gamma, \quad \text{if} \quad d \geq 3.
\] (2.5.11)

**Proof:** Suppose \((\tilde{f}, \tilde{p}) \in \overline{PE}_6\). Let us write \(f\) in local coordinates and also assume that \(\partial_x\) is the distinguished direction. Hence
\[
f = \frac{f_{03}}{6} y^3 + \frac{f_{40}}{24} x^4 + \ldots
\]
Note that
\[
f_{30}, \quad f_{21}, \quad 3f_{12}^2 - 4f_{21}f_{03} = 0.
\] (2.5.12)
Hence \((\tilde{f}, \tilde{p}) \in \overline{PD}_5^\gamma\), by Corollary 1.6.29. That proves the claim. □

**Lemma 2.5.6.** We have the following equality of sets
\[
\{(\tilde{f}, \tilde{p}) \in \overline{PA}_k : \Psi \overline{PD}_4(\tilde{f}, \tilde{p}) \neq 0\} = \overline{PA}_k \cup \overline{PA}_{k+1} \cup \{(\tilde{f}, \tilde{p}) \in \overline{PA}_{k+2} : \Psi \overline{PD}_4(\tilde{f}, \tilde{p}) \neq 0\}
\] (2.5.13)
if \(k \geq 3\) and \(d \geq k + 1\).

**Proof:** This follows from Proposition 1.6.24 and Lemma 1.6.1. □

**Lemma 2.5.7.** We have the following equality of sets
\[
\{(\tilde{f}, \tilde{p}) \in \overline{PD}_k : \Psi \overline{PE}_6(\tilde{f}, \tilde{p}) \neq 0\} = \overline{PD}_k \cup \overline{PD}_{k+1} \cup \{(\tilde{f}, \tilde{p}) \in \overline{PD}_{k+2} : \Psi \overline{PE}_6(\tilde{f}, \tilde{p}) \neq 0\}
\] (2.5.14)
if \(k \geq 6\) and \(d \geq k - 1\).

**Proof:** This follows from Proposition 1.6.37 and Lemma 1.6.1. □

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Bibliography


